

Lie 2-bialgebras*

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Abstract

In this paper, we study Lie 2-bialgebras, with special attention to coboundary ones, with the help of the cohomology theory of L_∞ -algebras with coefficients in L_∞ -modules. We construct examples of strict Lie 2-bialgebras from left-symmetric algebras and symplectic Lie algebras.

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1 Introduction

In this paper, we study the concept of a Lie 2-bialgebra with the hope to provide a certain categorification of the concept of a Lie bialgebra. We find a series of equations which can serve as higher classical Yang-Baxter equations, and give various solutions, thus naturally generating examples of Lie 2-bialgebras. For the construction of solutions, we use the tool of left-symmetric algebras (also known as pre-Lie algebras) and symplectic Lie algebras. The integration of Lie 2-bialgebra to a Quasi-Poisson Lie 2-group is studied in details in [12] .

A Lie bialgebra [17] is a pair of Lie algebra structures on dual vector spaces such that some compatibility conditions between them are satisfied. Lie bialgebras are the infinitesimal objects of Poisson-Lie groups. Both Lie bialgebras and Poisson-Lie groups are considered as semiclassical limits of quantum groups. Triangular quantum groups involve the solutions of quantum Yang-Baxter equations. In the classical limit, the solutions of classical Yang-Baxter equations provide examples of Lie bialgebras.

The categorification of Lie algebras are provided by 2-term L_∞ -algebras (they are also called Lie 2-algebras) [2]. The concept of L_∞ -algebras (sometimes called strongly homotopy (sh) Lie algebras) was originally introduced in [24, 32] as a model for “Lie algebras that satisfy Jacobi identity up to all higher homotopies”. The structure of Lie 2-algebras also appears widely in string theory, multisymplectic geometry [3, 4], and Courant algebroids [28, 29, 31]. Thus, to give a model for the categorification of Lie bialgebras, it is natural to consider a pair of Lie 2-algebra structures on dual 2-term complexes of vector spaces with some higher compatibility conditions, namely, a “2-term L_∞ -bialgebra”. But how does one allow homotopy inside a Lie bialgebra structure? A very nice method is explained by Kravchenko in [22] via higher derived brackets [1, 34] and Kosmann-Schwarzbach’s big bracket [20]: given a vector space V , we view the bracket $l \in \wedge^2 V^* \otimes V$ and cobracket $c \in V^* \otimes \wedge^2 V$ as elements in $\wedge^\bullet(V \oplus V^*)$, then a Lie bialgebra structure on (V, V^*) is equivalent to $\langle\langle l + c, l + c \rangle\rangle = 0$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is the big bracket defined by extending the natural pairing between V and V^* via the graded Leibniz rule:

$$\langle\langle v, u \wedge w \rangle\rangle = \langle\langle v, u \rangle\rangle \wedge w \pm u \wedge \langle\langle v, w \rangle\rangle.$$

Using this idea, Kravchenko then generalizes the above to a \mathbb{Z} -graded vector space V_\bullet and defines an L_∞ -bialgebra. However, in this setting, although a 2-term L_∞ -bialgebra gives a Lie 2-algebra structure on V , it does not give a Lie 2-algebra structure on the dual V^* . If one expects that a good categorification of Lie bialgebras should consist of Lie 2-algebra structures on V and V^* along with some compatibility conditions between them, then Kravchenko’s L_∞ -bialgebra needs to be modified. In [13], the authors nicely applied a simple shift to solve this problem. We adapt the shifting trick to our setting and have:

Definition 1.1. [Sect. 2.3] Let $V := V_0 \oplus V_{-1}$ be a 2-term \mathbb{Z} -graded vector space. A Lie 2-bialgebra structure on (V, V^*) is given by degree 4 elements $l_p \in \text{Sym}^{p,1}(V^*[-1] \otimes V[-2])$ and $c_q \in \text{Sym}^{1,q}(V^*[-1] \otimes V[-2])$ with $l_1 = c_1$, for $p, q \in \{1, 2, 3\}$, such that $\langle\langle \sum c_q + l_p, \sum c_q + l_p \rangle\rangle = 0$.

We observe that a first example of Lie 2-bialgebra is that of a Lie bialgebra \mathfrak{g} : which can be viewed as a Lie 2-bialgebra $\mathfrak{g} \oplus \mathfrak{g}$ (see Remark 3.9).

Proposition 1.2. *If $(V, V^*, l_1 = c_1, l_2, l_3, c_2, c_3)$ is a Lie 2-bialgebra, then both (V, l_1, l_2, l_3) and $(V^*[1], c_1, c_2, c_3)$ are Lie 2-algebras.*

Thus, l_p 's and c_q 's are understood as brackets and cobrackets respectively. Furthermore, we find the compatibility conditions between brackets and cobrackets:

Theorem 1.3. [Sect. 2.3] *Lie 2-algebras $(V := V_0 \oplus V_{-1}, l_1, l_2, l_3)$ and $(V^*[1], c_1, c_2, c_3)$ form a Lie 2-bialgebra if and only if $l_1 = c_1$ as elements in $V^* \oplus V$, and $\sum c_q$ is a 4-cocycle representing an element in $H^4(V, \text{Sym}(V[-2]))$.*

For the purpose of this theorem, we develop the cohomology theory of an L_∞ -algebra L with coefficients in representation on a k -term complex of vector spaces (known as L_∞ -modules). When we restrict to adjoint representation we recover the cohomology studied in [26]. We give the adjoint representation of L in terms of big brackets. We also introduce Manin triples in this general framework. Here we see the advantage of the language of big brackets: it makes concepts and calculations very elegant and intrinsic. However, it also has the disadvantage that under some circumstances it is not explicit enough to give examples. So we then focus on the strict case in Sect. 2.4, and explain the above concepts in concrete formulas familiar to general algebraists. Associated to any k -term complex of vector spaces \mathcal{V} , there is a natural differential graded Lie algebra [23, 30], which plays the same role as $\mathfrak{gl}(V)$ for a vector space V in the classical case. In the strict case, it is enough to look at the strict Lie 2-algebra $\text{End}(\mathcal{V})$ obtained by truncation. This simplification makes it possible to write down concrete formulas. This is the content of Section 2.

The above leads to our study of strict Lie 2-bialgebras in Section 3. In fact, this section in the article is written in the usual algebraic language without various grades and shifts, and it is self-contained. Thus, readers who find the L_∞ -language confusing can simply skip Section 2 and directly go to Section 3. Guided by the classical theory of Lie bialgebras, we explore, in explicit terms, various higher corresponding objects—matched pairs, Manin triples, standard Manin triples—and their relations. We first study the conditions under which the direct sum of two strict Lie 2-algebras with representations on each other is a strict Lie 2-algebra (Theorem 3.1). Two strict Lie 2-algebras with representations on each other satisfy these conditions is called a matched pair. For a strict Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ and a 2-cocycle (δ_0, δ_1) , where $\delta_0 \in \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1})$ and $\delta_1 \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1})$ defining a semidirect product Lie algebra structure on $\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^*$, then $\mathfrak{g}^* = (\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*, d^*, [\cdot, \cdot]^*)$ is a strict Lie 2-algebra such that $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*)$ is a matched pair (Theorem 3.7). Furthermore, in Sect. 3.4, we focus on the coboundary case, i.e., we require (δ_0, δ_1) to be an exact 2-cocycle. Due to the abundant content of the corresponding cohomology theory, we find that there are more generalized r -matrices than in the classical case. We work out a set of higher classical Yang-Baxter equations (higher CYBE) whose solutions provide examples of Lie 2-bialgebras.

In Section 4, we construct various (coboundary) strict Lie 2-bialgebras via explicit solutions of higher CYBE given by left-symmetric algebras. Left-symmetric algebras (or pre-Lie algebras, Vinberg algebras, and etc.) arose from the study of affine manifolds and affine Lie groups [21], convex homogeneous cones [33] and deformations of associative algebras [18] and then appeared in

many fields in mathematics and mathematical physics (see the survey article [10] and the references therein). In particular, there is a close relationship between left-symmetric algebras and classical Yang-Baxter equation, which leads to regard the former as the algebraic structures behind the latter [6]. We use the classification of low dimensional left-symmetric algebras and give an explicit example of a low dimensional Lie 2-bialgebra (Example 4.9).

Furthermore, left-symmetric algebras are also regarded as the underlying algebraic structures of symplectic Lie algebras [14], which coincides with Drinfeld's observation of the correspondence between the invertible (skew-symmetric) classical r -matrices and the symplectic forms on Lie algebras [16]. We then construct a general type of Lie 2-bialgebras arising naturally from symplectic Lie algebras (Example 4.15). The naturality of the construction suggests some geometric meaning of such Lie 2-bialgebras, which is however still a mystery to us.

Finally, since Lie bialgebras can be viewed as semiclassical limits of quantum groups, a natural question to ask is whether there is some relation possibly via quantization between our categorification and Khovanov-Lauda's recent categorification of quantum groups [19]. At this very early stage, as far as we can tell, the two sorts of categorification are rather different. Any relation if existed will be nontrivial to establish. Also, we do not claim our work is the final word of categorification of Lie bialgebras with respect to the above aspect. Instead, we regard it as something which opens a rather interesting direction, along which we are currently working.

General comments: in this paper, our vector spaces, when not specified otherwise, are all finite dimensional and over a base field \mathbb{k} with characteristic 0. In fact, for us the most important applications are cases where either $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$.

Notations: DGLA is short for differential graded Lie algebras; x, y, z are arbitrary elements in \mathfrak{g}_0 ; x^*, y^*, z^* are arbitrary elements in \mathfrak{g}_0^* ; h, k, l are arbitrary elements in \mathfrak{g}_{-1} and h^*, k^*, l^* are arbitrary elements in \mathfrak{g}_{-1}^* ; for a graded vector space $V = \sum_n V_n$, $V[l]$ denote the l -shifted graded vector space, namely $V[l]_n = V_{l+n}$; $Sym(V)$ is the symmetric power of V .

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2 Lie 2-bialgebras

2.1 Lie 2-algebras via L_∞ -algebras

Lie algebras can be categorified to Lie 2-algebras. For a good introduction on this subject see [2].

Definition 2.1. An L_∞ -algebra is a graded vector space $L = L_0 \oplus L_{-1} \oplus \cdots$ equipped with a system $\{l_k \mid 1 \leq k < \infty\}$ of linear maps $l_k : \wedge^k L \rightarrow L$ with degree $\deg(l_k) = 2 - k$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign "Ksgn" is satisfied for all $n \geq 0$:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \quad (1)$$

where the summation is taken over all $(i, n-i)$ -unshuffles with $i \geq 1$. A **Lie 2-algebra** is a 2-term L_∞ -algebra $L = L_0 \oplus L_{-1}$.

A 1-term L_∞ -algebra L_0 is a Lie algebra in the usual sense. A 2-term L_∞ -algebra (a Lie 2-algebra) has only possibly nonzero brackets l_1 , l_2 , and l_3 . The compatibility condition (1) tells that l_1 is a derivation with respect to l_2 , and l_3 controls the obstruction of Jacobi identity of l_2 . A **strict Lie 2-algebra** is a Lie 2-algebra with $l_3 = 0$. This specifically tells us that, without any gradings, a strict Lie 2-algebra \mathfrak{g} is simply a complex of (non-graded usual) vector spaces $\mathfrak{g}_{-1} \xrightarrow{l_1=d} \mathfrak{g}_0$ equipped with a graded Lie bracket $l_2 = [\cdot, \cdot] : \mathfrak{g}_i \times \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}$, where $-1 \leq i + j \leq 0$, such that for all $x, y, z \in \mathfrak{g}_0$ and $h, k \in \mathfrak{g}_{-1}$, we have

$$\begin{aligned} [x, y] &= -[y, x], [x, h] = -[h, x], [h, k] = 0, \quad d([x, h]) = [x, dh], [dh, k] = [h, dk], \\ [[x, y], z] + [[y, z], x] + [[z, x], y] &= 0, \quad [[x, y], h] + [[y, h], x] + [[h, x], y] = 0. \end{aligned} \quad (2)$$

The notions of a strict Lie 2-algebra, a crossed module of Lie algebras, and a 2-term DGLA are equivalent. Moreover, Eq. (2) implies that there is a Lie algebra structure (semidirect product) on $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$:

$$[x + h, y + k]_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}} = [x, y] + [x, k] + [h, y].$$

such that d is a graded derivation with respect to $[\cdot, \cdot]$.

An L_∞ -algebra L gives a differential graded commutative algebra (d.g.c.a) structure on graded symmetric algebra

$$Sym(L^*[-1]) = \underbrace{\mathbb{k}}_{\text{degree 0}} \oplus \underbrace{L_0^*}_{\text{degree 1}} \oplus \underbrace{[\wedge^2 L_0^* \oplus L_{-1}^*]}_{\text{degree 2}} \oplus \underbrace{[\wedge^3 L_0^* \oplus L_0^* \otimes L_{-1}^* \oplus L_{-2}^*]}_{\text{degree 3}} \oplus \dots,$$

whose degree 1 differential δ is given by dualizing l_i 's¹. In fact, the generalized Jacobi identity (1) is equivalent to $\delta^2 = 0$. Then an L_∞ -morphism $f : L \rightarrow L'$ is given by a d.g.c.a. morphism $Sym((L')^*[-1]) \rightarrow Sym(L^*[-1])$. It is more general than a “strict morphism”, namely a morphism preserves all brackets strictly. More precisely,

Definition 2.2. [2] Let \mathfrak{g} and \mathfrak{g}' be two strict Lie 2-algebras. A **strict homomorphism** μ from \mathfrak{g} to \mathfrak{g}' consists of linear maps $\mu_0 : \mathfrak{g}_0 \rightarrow \mathfrak{g}'_0$ and $\mu_1 : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}'_{-1}$ commuting with the differential, i.e.,

$$\mu_0 \circ d = d' \circ \mu_1,$$

such that

$$\begin{cases} [(\mu_0(x), \mu_0(y))' - \mu_0[x, y]] &= 0, \\ [\mu_0(x), \mu_1(h)]' - \mu_1[x, h] &= 0. \end{cases} \quad (3)$$

2.2 L_∞ -modules and L_∞ -cohomology

Now we recall the definition of L_∞ -modules [23]. Given a k -term complex of vector spaces $\mathcal{V} : V_{-k+1} \xrightarrow{\partial} \dots V_{-1} \xrightarrow{\partial} V_0$, the endomorphisms (not necessarily preserving the degree) form a DGLA $\mathfrak{gl}(\mathcal{V})$ with the graded commuting bracket and a differential inherited from ∂ . It plays the same role as $\mathfrak{gl}(V)$ in the classical case for a vector space V (see [23, 30] for details). Thus an L_∞ -algebra L naturally represents on a complex of vector space \mathcal{V} by an L_∞ -morphism $L \rightarrow \mathfrak{gl}(\mathcal{V})$, in this case \mathcal{V} is called an L_∞ -**module** of L .

¹This works for finite dimensional L_i 's, which is our setting in this paper.

Another **equivalent definition of an L_∞ -module**² of L is given via a generalized Chevalley-Eilenberg complex of L , that is, an L_∞ -module structure on a graded vector space \mathcal{V} is given by a degree one differential D on the graded vector space

$$(Sym(L^*[-1]) \otimes \mathcal{V})_n = \oplus_k Sym(L^*[-1])_k \otimes V_{n-k}.$$

The second definition is subtly different from the first definition in that in the second definition, via D , L directly represents on a graded vector space \mathcal{V} and D also encodes the differential on \mathcal{V} . However, this small difference does not affect our application. This way of defining L_∞ -modules has the advantage that we can view $(Sym(L^*[-1]) \otimes \mathcal{V}, D)$ as the cochain complex $C^\bullet(L, \mathcal{V})$ of L with coefficient in its L_∞ -module \mathcal{V} . Then its cohomology group $H^\bullet(L, \mathcal{V})$ is defined to be the **L_∞ -algebra cohomology** of L with coefficient \mathcal{V} . We denote the set of L_∞ -modules of L by $Rep^\infty(L)$ and a typical element in $Rep^\infty(L)$ by $(\mathcal{V}, D_\mathcal{V})$.

Given $(\mathcal{V}, D_\mathcal{V}), (\mathcal{W}, D_\mathcal{W}) \in Rep^\infty(L)$, there is a degree 1 differential D on $Sym(L^*[-1]) \otimes (\mathcal{V} \otimes \mathcal{W})$ uniquely determined by

$$D(\eta \otimes \xi) = D_\mathcal{V}(\eta) \otimes \xi + (-1)^{|\eta|} \eta \otimes D_\mathcal{W}(\xi),$$

for all $\eta \in Sym(L^*[-1]) \otimes \mathcal{V}$ and $\xi \in Sym(L^*[-1]) \otimes \mathcal{W}$.

Similarly, one can take the symmetric power $Sym(\mathcal{V})$, wedge product $\Lambda(\mathcal{V})$, and dual \mathcal{V}^* of L_∞ -modules.

2.3 L_∞ -bialgebras

- Big bracket:

Given a graded vector space $V = \sum_n V_n$, it is well known that there is a graded version of big bracket $\langle\langle \cdot, \cdot \rangle\rangle$ on $Sym(V^*[k]) \otimes Sym(V[l]) \cong Sym(V^*[k] \oplus V[l]) \cong Sym(T^*[l+k]V[l])$ by extending the usual pairing between V^* and V via a graded Leibniz rule³

$$\begin{aligned} \langle\langle u, v \wedge w \rangle\rangle &= \langle\langle u, v \rangle\rangle \wedge w + (-1)^{(|u|+l+k)|v|} v \wedge \langle\langle u, w \rangle\rangle, \\ \langle\langle u, v \rangle\rangle &= -(-1)^{(|u|+k+l)(|v|+k+l)} \langle\langle v, u \rangle\rangle, \end{aligned} \tag{4}$$

where $u \in Sym(V^*[k] \oplus V[l])_{|u|}$ and $v \in Sym(V^*[k] \oplus V[l])_{|v|}$ respectively. The big bracket is in fact the canonical graded Poisson bracket on $T^*[l+k]V[l]$. Thus, in addition, we have a graded Jacobi identity:

$$\langle\langle u, \langle\langle v, w \rangle\rangle \rangle\rangle = \langle\langle \langle\langle u, v \rangle\rangle, w \rangle\rangle + (-1)^{(|u|+k+l)(|v|+k+l)} \langle\langle v, \langle\langle u, w \rangle\rangle \rangle\rangle. \tag{5}$$

- L_∞ -algebras via big bracket:

Given an L_∞ -algebra L , the bracket l_i can be viewed as a degree 2 element in $Sym(L^*[-1]) \otimes L$, for example:

$$l_2 : \wedge^2 L_0 \rightarrow L_0, \quad \rightsquigarrow \quad l_2|_{\wedge^2 L_0} \in \wedge^2 L_0^* \otimes L_0 \subset Sym^2(L^*[-1]) \otimes L.$$

In fact, l_i can be viewed as a degree $l - k + 2$ element in $(Sym(L^*[-1]) \otimes L[k])[l]$.

²The equivalence is supposed to be well known. For a detailed proof, we refer to [15].

³Here we specially thank Yvette Kosmann-Schwarzbach, Rajan Mehta and Dmitry Roytenberg for their help on the signs and on the history of various brackets [25].

Lemma 2.3. *A series of degree $2-l$ elements $l_i \in (Sym^i(V^*[-1]) \otimes V[l])$ with $i = 1, 2, \dots$ on $V = \sum_{n=0}^{-\infty} V_n$ gives an L_∞ -structure if and only if $\langle\langle \sum_i l_i, \sum_i l_i \rangle\rangle = 0$.*

Proof. The proof depends on a simple observation: there is a degree 1 operator δ on $Sym(V^*[-1])$ given by

$$Sym(V^*[-1]) \xrightarrow{\delta := \langle\langle \sum l_i, \cdot \rangle\rangle} Sym(V^*[-1]).$$

The fact that δ is a derivation follows from the graded Leibniz rule (4) of $\langle\langle \cdot, \cdot \rangle\rangle$. Then $\delta^2 = 0$ iff l_i 's form an L_∞ -structure on V . Then by graded Jacobi identity, we have

$$\langle\langle \sum l_i, \langle\langle \sum l_i, u \rangle\rangle \rangle\rangle = \langle\langle \sum l_i, \sum l_i \rangle\rangle u + (-1)^{1 \cdot 1} \langle\langle \sum l_i, \langle\langle \sum l_i, u \rangle\rangle \rangle\rangle,$$

which implies $2\delta^2 u = \langle\langle \sum l_i, \sum l_i \rangle\rangle u$. Finally, δ has degree 1 by Lemma 2.4. ■

Lemma 2.4. *Given an element $t \in Sym(V^*[k] \oplus V[l])$, the degree of the operator $\langle\langle t, \cdot \rangle\rangle : Sym(V^*[k] \oplus V[l]) \rightarrow Sym(V^*[k] \oplus V[l])$ is $|t| + k + l$.*

Proof. Straightforward calculations. ■

• Lie 2-bialgebras via L_∞ -bialgebras:

A similar theory holds for the Lie bialgebra setting, that is, an L_∞ -bialgebra also corresponds to a d.g.c.a. but with the differential $\delta = \langle\langle \sum t_{pq}, \cdot \rangle\rangle$ coming from more complicated data including brackets $t_{p1} \in (Sym^p(V^*[-1]) \otimes V[-1])$, cobrackets $t_{1p} \in (V^*[-1] \otimes Sym^p(V[-1]))$ and their relations $t_{pq} \in (Sym^p(V^*[-1]) \otimes Sym^q(V[-1]))$ for $p, q \geq 2$. Here with various degree shiftings, t_{pq} 's have degree 1. This is equivalent to requiring their total degree (without shifting) to be 1 as in [22] (see also Remark 2.5). Nothing stops us from shifting further to adapt the notion to our application, which leads to Definition 1.1. In fact, with the above interpretation, since the Lie 2-bialgebra we define in Definition 1.1 does not contain t_{22} -term, it is *relation-wise strict* in the sense that the brackets and cobrackets satisfy strict relations.

Remark 2.5. In our Definition 1.1 of Lie 2-bialgebras, L_{-1}^* is of degree 0 and L_0^* is of degree -1 in the Lie 2-algebra $L_0^* \xrightarrow{d^*} L_{-1}^*$. This is not the same as Kravchenko's convention on degrees [22], where for an L_∞ -algebra V , the total degree of an element $\xi_1 \cdots \xi_p \alpha_1 \cdots \alpha_q$ in $\wedge^p V^* \otimes \wedge^q V$ is defined to be $\sum_{i=1}^p |\xi_i| + \sum_{i=1}^q |\alpha_i| + p + q - 2$. Let us explain this in the 2-term case. The L_∞ -coalgebra structure in Kravchenko's setting is given by maps $\gamma_p : V \rightarrow \wedge^p V$ of total degree 1. In particular, γ_1 is a map from V_{-1} to V_0 ; γ_2 is an element in $V_0^* \otimes \wedge^2 V_0 \oplus V_{-1}^* \otimes V_0 \wedge V_{-1}$. So what one obtains is not the usual 2-term L_∞ -algebra structure on $L_0^* \xrightarrow{\gamma_1^*} L_{-1}^*$. Our degree shifting trick allows us to adjust the map γ_2 to obtain the usual 2-term L_∞ -algebra structure on $L_0^* \xrightarrow{\gamma_1^*} L_{-1}^*$. In terms of total degrees, we define the total degree of an element $\xi_1 \cdots \xi_p \alpha_1 \cdots \alpha_q$ in $\wedge^p V^* \otimes \wedge^q V$ to be $\sum_{i=1}^p |\xi_i| + \sum_{i=1}^q |\alpha_i| + p + 2q - 3$. Then the t_{pq} 's will have total degree 1 in a non-shifted complex.

Lemma 2.6. *A series of degree 4 elements $l_i \in (Sym^i(V^*[-1]) \otimes V[-2])$ and $c_i \in (V^*[-1] \otimes Sym^i(V[-2]))$ with $i = 1, 2, \dots$ on $V = V_0 \oplus V_{-1}$ gives a Lie 2-bialgebra structure if and only if $c_1 = l_1$ and $\delta := \langle\langle \sum_i l_i + c_i, \cdot \rangle\rangle$ defines a d.g.c.a. on $(Sym(V^*[-1]) \otimes Sym(V[-2]))$.*

Proof. Given such degree 4 elements l_i and c_i on V , by degree reason $l_i = c_i = 0$ for $i \geq 3$. Moreover, by Lemma 2.4, δ has degree 1 = 4 - 3. Then degree 4 elements l_i and c_i with $i = 1, 2, 3$ define a Lie 2-bialgebra structure on V if and only if $l_1 = c_1$ and $\langle\langle \sum l_i + c_i, \sum l_i + c_i \rangle\rangle = 0$. Then

the rest of the proof follows from the graded Jacobi identity (5) and the graded Leibniz rule (4) as in Lemma 2.3. ■

Now we are ready to give the proof of Proposition 1.2:

Proof of Proposition 1.2. By degree reason, $\langle\langle \sum l_i + c_i, \sum l_i + c_i \rangle\rangle = 0$ is equivalent to

$$\begin{aligned} \langle\langle l_1, l_1 \rangle\rangle &= 0, & \langle\langle c_1, c_2 \rangle\rangle &= 0, & \langle\langle l_1, l_2 \rangle\rangle &= 0, \\ \langle\langle l_2, l_2 \rangle\rangle + 2\langle\langle l_3, l_1 \rangle\rangle &= 0, & \langle\langle l_2, l_3 \rangle\rangle &= 0, & \langle\langle c_2, c_2 \rangle\rangle + 2\langle\langle c_3, c_1 \rangle\rangle &= 0, & \langle\langle c_2, c_3 \rangle\rangle &= 0, \\ \langle\langle l_2, c_2 \rangle\rangle &= 0, & \langle\langle l_2, c_3 \rangle\rangle &= 0, & \langle\langle l_3, c_2 \rangle\rangle &= 0, & \langle\langle l_3, c_3 \rangle\rangle &= 0 \end{aligned} \quad (6)$$

The first two lines of equations implies that $\langle\langle \sum l_i, \sum l_i \rangle\rangle = 0$ and $\langle\langle \sum c_i, \sum c_i \rangle\rangle = 0$. Notice that $\sum l_i \in \text{Sym}(L^*[-1]) \otimes L[-2]$ and $\sum c_i \in \text{Sym}^1(L^*[-1]) \otimes \text{Sym}(L[-2]) = \text{Sym}((L^*[1])^*[-1]) \otimes (L^*[1])[-2]$. By Lemma 2.3, the above two conditions are equivalent to the fact that l_i 's and c_i 's give L_∞ -structures on L and $L^*[1]$ respectively. ■

Given an L_∞ -algebra L , it is easy to describe its adjoint representation on any shift $L[k]$ via the big bracket:

$$D_{ad} := \langle\langle \sum l_i, \cdot \rangle\rangle : \text{Sym}(L^*[-1]) \otimes L[k] \rightarrow \text{Sym}(L^*[-1]) \otimes L[k].$$

This extends to symmetric algebras,

$$\tilde{D}_{ad} := \langle\langle \sum l_i, \cdot \rangle\rangle : \text{Sym}(L^*[-1]) \otimes \text{Sym}(L[k]) \rightarrow \text{Sym}(L^*[-1]) \otimes \text{Sym}(L[k]). \quad (7)$$

To justify that D_{ad} and \tilde{D}_{ad} indeed define a representation, we need the following lemma,

Lemma 2.7. *With the above notation, we have $D_{ad}^2 = 0$ and $\tilde{D}_{ad}^2 = 0$.*

Proof. Since l_i 's are L_∞ -brackets for L , by Lemma 2.3, $\sum l_i$ are degree $2-k$ elements in $\text{Sym}(L^*[-1]) \otimes L[k]$ thus in $\text{Sym}(L^*[-1]) \otimes \text{Sym}(L[k])$, moreover $\langle\langle \sum l_i, \sum l_i \rangle\rangle = 0$. Thus

$$\langle\langle \sum l_i, \langle\langle \sum l_i, u \rangle\rangle \rangle\rangle = \langle\langle \sum l_i, \sum l_i \rangle\rangle u + (-1)^{(2-k-1+k)^2} \langle\langle \sum l_i, \langle\langle \sum l_i, u \rangle\rangle \rangle\rangle,$$

which implies both $D_{ad}^2 = 0$ and $\tilde{D}_{ad}^2 = 0$. ■

Now we are ready to give the following proof:

Proof of Theorem 1.3. Given a Lie 2-bialgebra L , the sum of the cobrackets $\sum_{i=1}^3 c_i \in (\text{Sym}^1(L^*[-1]) \otimes \text{Sym}(L[-2]))$ being a degree 4 element, is a 4-cochain in $C^4(L, \text{Sym}(L[-2]))$. The last line of Eq. (6) is equivalent to $\langle\langle l_2 + l_3, c_2 + c_3 \rangle\rangle = 0$. If $\langle\langle l_1, l_1 \rangle\rangle = 0$, (which is the case when L is a Lie 2-algebra), we have further more $\langle\langle l_1 + l_2 + l_3, c_1 + c_2 + c_3 \rangle\rangle = 0$, keeping in mind that $l_1 = c_1$. This is equivalent to $\tilde{D}_{ad}(\sum c_i) = 0$. ■

- Manin triples:

As in the classical case, we have yet another description of L_∞ -bialgebras via Manin $L_\infty[k]$ -triples. When $k = 0$, we obtain [22, Def. 32].

Definition 2.8. A Manin $L_\infty[k]$ -triple is a triple of L_∞ -algebras $(\mathfrak{g}, \mathfrak{g}^+, \mathfrak{g}^-)$ equipped with a nondegenerate graded symmetric bilinear form $S(\cdot, \cdot)$, such that

1. $S(\cdot, \cdot)$ has degree k , that is, there is an identification of $(\mathfrak{g}^+)^*$ with $\mathfrak{g}^-[k]$ via $S(\cdot, \cdot)$.
2. $\mathfrak{g}^+, \mathfrak{g}^-$ are L_∞ -subalgebras of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ as graded vector spaces;
3. \mathfrak{g}^+ and \mathfrak{g}^- are isotropic with respect to $S(\cdot, \cdot)$;
4. the n -brackets λ_n of the L_∞ -algebra structure on \mathfrak{g} are invariant with respect to $S(\cdot, \cdot)$, that is,

$$S(\lambda_n(a_1, \dots, a_n), a_0) = (-1)^{|a_n||a_0|} S(\lambda_n(a_1, \dots, a_{n-1}, a_0), a_n).$$

Then a 2-term Manin $L_\infty[1]$ -triple is a Manin $L_\infty[k]$ -triple $(\mathfrak{g}, \mathfrak{g}^+, \mathfrak{g}^-)$ such that all three L_∞ -algebras have only 2-terms. We will explain this in more concrete terms in Section 3.2.

Then as expected, we have:

Theorem 2.9. *The notions of Lie 2-bialgebra and a 2-term Manin $L_\infty[1]$ -triple are equivalent.*

Proof. The proof is done by adding some careful counting of degrees to the proof of [22, Thm. 33]. We refer to [15] for this treatment. ■

2.4 Strict case

Now we explain the abstract definitions given in prior sections with explicit formulas in the case of a strict Lie 2-algebra. This is a preparation for the next section, where we address strict Lie 2-bialgebras in a more classical setting. It is not redundant because with the concrete picture, we will address the non-symmetric version to make it better connected to the usual algebraic discussion of Lie bialgebras. What is important for us is the strict Lie 2-algebra of the 2-truncation of the endomorphism DGLA $\mathfrak{gl}(\mathcal{V})$ because an L_∞ -morphism $L \rightarrow \mathfrak{gl}(\mathcal{V})$ can only see this part. We denote the truncation by $\text{End}(\mathcal{V})$,

$$\text{End}(\mathcal{V}) : \text{End}^{-1}(\mathcal{V}) \xrightarrow{\delta} \text{End}_\partial^0(\mathcal{V}). \quad (8)$$

We describe it in explicit terms: $\text{End}^{-1}(\mathcal{V})$ is the maximal subspace E of $\bigoplus_{i=-k+2}^0 \text{End}(V_i, V_{i-1})$ satisfying $[E, E]_C = 0$, where $[\cdot, \cdot]_C$ is the natural commutator, i.e.,

$$[(\tau_{k-2}, \dots, \tau_0), (\varsigma_{k-2}, \dots, \varsigma_0)]_C = (\tau_{k-2} \circ \varsigma_{k-3} - \varsigma_{k-2} \circ \tau_{k-3}, \dots, \tau_1 \circ \varsigma_0 - \varsigma_1 \circ \tau_0),$$

for any $\tau = (\tau_{k-2}, \dots, \tau_0), \varsigma = (\varsigma_{k-2}, \dots, \varsigma_0) \in \bigoplus_{i=-k+2}^0 \text{End}(V_i, V_{i-1})$. Then $\text{End}_\partial^0(\mathcal{V})$ is given by

$$\text{End}_\partial^0(\mathcal{V}) = \{A = (A_{k-1}, \dots, A_1, A_0) \in \bigoplus_{i=-k+1}^0 \text{End}(V_i) \mid A_j \circ \partial = \partial \circ A_{j+1}, \forall j\}.$$

The strict Lie 2-algebra structure is given by

$$\begin{aligned} \delta(\tau) &= \partial \circ \tau + \tau \circ \partial, \\ [A, B] &= A \circ B - B \circ A = ([A_{k-1}, B_{k-1}], \dots, [A_1, B_1]), \\ [A, \tau] &= A \circ \tau - \tau \circ A = (A_{k-1} \circ \tau_{k-2} - \tau_{k-2} \circ A_{k-2}, \dots, A_1 \circ \tau_0 - \tau_0 \circ A_0), \end{aligned}$$

for any $A = (A_{k-1}, \dots, A_0), B = (B_{k-1}, \dots, B_0) \in \text{End}_\partial^0(\mathcal{V})$, and $\tau = (\tau_{k-2}, \dots, \tau_0) \in \text{End}^{-1}(\mathcal{V})$.

Definition 2.10. A **strict representation** of a strict Lie 2-algebra \mathfrak{g} on a k -term complex of vector spaces \mathcal{V} is a strict homomorphism $\mu = (\mu_0, \mu_1)$ from \mathfrak{g} to the strict Lie 2-algebra $\text{End}(\mathcal{V})$. We denote a strict representation by $(\mathcal{V}; \mu)$.

Remark 2.11. Such a strict representation of \mathfrak{g} clearly is a special L_∞ -module of \mathfrak{g} viewed as an L_∞ -algebra.

Let $(\mathcal{V}; \mu)$ be a k -term strict representation of \mathfrak{g} . To obtain the cohomology $H^\bullet(\mathfrak{g}, \mathcal{V})$, the space of p -cochains is $C^p(\mathfrak{g}, \mathcal{V}) := (\text{Sym}(\mathfrak{g}^*[-1]) \otimes \mathcal{V})_p$. The differential operator D can be explicitly written as

$$D = \widehat{d} + d_\mu + \widehat{\partial} : C^p(\mathfrak{g}, \mathcal{V}) \rightarrow C^{p+1}(\mathfrak{g}, \mathcal{V}).$$

We explain each term explicitly: $\widehat{d} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_s) \longrightarrow \text{Hom}(\wedge^{p-1} \mathfrak{g}_0 \wedge \wedge^{q+1} \mathfrak{g}_{-1}, V_s)$ is the degree 1 operator induced by d :

$$\begin{aligned} & \widehat{d}(f)(x_1, \dots, x_{p-1}, h_1, h_2, \dots, h_{q+1}) \\ &= (-1)^{p+q} (f(x_1, \dots, x_{p-1}, dh_1, h_2, \dots, h_{q+1}) + c.p.(h_1, \dots, h_{q+1})), \end{aligned}$$

and $\widehat{\partial} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_s) \longrightarrow \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_{s+1})$ is the degree 1 operator induced by ∂ :

$$\widehat{\partial}(f) = (-1)^{p+2q} \partial \circ f, \quad (9)$$

and $d_\mu = (d_\mu^{(1,0)}, d_\mu^{(0,1)})$, where $d_\mu^{(1,0)} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_s) \longrightarrow \text{Hom}(\wedge^{p+1} \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_s)$ is given by

$$\begin{aligned} & d_\mu^{(1,0)}(f)(x_1, \dots, x_{p+1}, h_1, \dots, h_q) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \mu_0(x_i) f(x_1, \dots, \widehat{x}_i, \dots, x_{p+1}, h_1, \dots, h_q) \\ & \quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{p+1}, h_1, \dots, h_q), \\ & \quad + \sum_{i,j} (-1)^i f(x_1, \dots, \widehat{x}_i, \dots, x_{p+1}, h_1, \dots, [x_i, h_j], \dots, h_q), \end{aligned}$$

and $d_\mu^{(0,1)} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^q \mathfrak{g}_{-1}, V_s) \longrightarrow \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \wedge^{q+1} \mathfrak{g}_{-1}, V_{s-1})$ is given by

$$d_\mu^{(0,1)}(f)(x_1, \dots, x_p, h_1, \dots, h_{q+1}) = \sum_{i=1}^{q+1} (-1)^{p+i+1} \mu_1(h_i) f(x_1, \dots, x_p, h_1, \dots, \widehat{h}_i, \dots, h_{q+1}).$$

For any strict representation $(\mathcal{V}; \mu)$ of \mathfrak{g} , let $\mathcal{V}^* : V_0^* \xrightarrow{\partial^*} V_{-1}^* \xrightarrow{\partial^*} \dots V_{-k+1}^*$ be the dual complex of \mathcal{V} . The dual representation $\mu_0^* : \mathfrak{g}_0 \longrightarrow \bigoplus_{i=-k+1}^0 \text{End}(V_i^*)$ and $\mu_1^* : \mathfrak{g}_{-1} \longrightarrow \text{End}^{-1}(\mathcal{V}^*)$ is defined by

$$\begin{aligned} \langle \mu_0^*(x) u^*, v \rangle &= -\langle u^*, \mu_0(x) v \rangle, \quad \forall u^* \in V_i^*, v \in V_i, \\ \langle \mu_1^*(h) p^*, u \rangle &= -\langle p^*, \mu_1(h) u \rangle, \quad \forall p^* \in V_i^*, u \in V_{i+1}. \end{aligned}$$

In fact, it is straightforward to see that μ_0^* commutes with ∂^* , i.e., $\mu_0^* \in \text{End}_{\partial^*}^0(\mathcal{V}^*)$. Furthermore, $\mu^* \triangleq (\mu_0^*, \mu_1^*)$ is a strict homomorphism from \mathfrak{g} to $\text{End}(\mathcal{V}^*)$.

If both $(\mathcal{V}; \mu^V)$ and $(\mathcal{W}; \mu^W)$ are strict representations of \mathfrak{g} , then the tensor product $(\mathcal{V} \otimes \mathcal{W}; \mu)$ is also a strict representation of \mathfrak{g} , where $\mu = (\mu_0, \mu_1)$ is explicitly given by

$$\mu_0 = \mu_0^V \otimes 1 + 1 \otimes \mu_0^W, \quad \mu_1 = \mu_1^V \otimes 1 + 1 \otimes \mu_1^W.$$

The adjoint representation of \mathfrak{g} on itself, denote by $\text{ad} = (\text{ad}_0, \text{ad}_1)$, with,

$$\begin{aligned} \text{ad}_0(x) &= [x, \cdot] \in \text{End}_d^0(\mathfrak{g}), \quad \forall x \in \mathfrak{g}_0, \\ \text{ad}_1(h) &= [h, \cdot] \in \text{End}^1(\mathfrak{g}), \quad \forall h \in \mathfrak{g}_{-1}, \end{aligned}$$

is a strict representation. The dual representation of \mathfrak{g} on \mathfrak{g}^* is called the **coadjoint representation** and denoted by $\text{ad}^* = (\text{ad}_0^*, \text{ad}_1^*)$. Then \mathfrak{g} acts on $\mathfrak{g}[-1] \otimes \mathfrak{g}[-1]$ —a 3-term complex of vector spaces

$$(\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0 := \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1} \xrightarrow{d^\otimes} (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1 := \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \xrightarrow{d^\otimes} (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_2 := \mathfrak{g}_0 \otimes \mathfrak{g}_0, \quad (10)$$

with d^\otimes given by

$$\begin{aligned} d^\otimes(h \otimes k) &= (d \otimes 1 + 1 \otimes d)(h \otimes k) = dh \otimes k + h \otimes dk, \quad h, k \in \mathfrak{g}_{-1}, \\ d^\otimes(x \otimes k + h \otimes y) &= (d \otimes 1 - 1 \otimes d)(x \otimes k + h \otimes y) = dh \otimes y - x \otimes dk, \quad x, y \in \mathfrak{g}_0. \end{aligned}$$

This representation plays an essential role when we consider the strict Lie 2-bialgebras: thanks to the strict setting, we do not have to work with the whole $\text{Sym}(\mathfrak{g})$ as we have done in the nonstrict case (see also Remark 3.5 for the different degree shifting). The corresponding Chevalley-Eilenberg complex is given by

$$\begin{aligned} &(\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0 \xrightarrow{D} (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1 \oplus \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \xrightarrow{D} \\ &(\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_2 \oplus \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1) \oplus \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \oplus \\ &\text{Hom}(\wedge^2 \mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \xrightarrow{D} \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_2) \oplus \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1) \oplus \\ &\text{Hom}(\wedge^2 \mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1) \oplus \text{Hom}(\wedge^3 \mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \oplus \text{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \\ &\xrightarrow{D} \dots, \end{aligned} \quad (11)$$

where $D = \widehat{d} + d_{\text{ad}} + \widehat{d}^\otimes$, in which d_{ad} is the operator associated to the tensor representation $(\text{ad}_0 \otimes 1 + 1 \otimes \text{ad}_0, \text{ad}_1 \otimes 1 + 1 \otimes \text{ad}_1)$ of \mathfrak{g} on $\mathfrak{g}[-1] \otimes \mathfrak{g}[-1]$. For a 2-cochain $(\delta_0, \delta_1) \in \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1) \oplus \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0)$, we have

$$D(\delta_0, \delta_1) = -d^\otimes \circ \delta_0 + d_{\text{ad}} \delta_0 - \delta_0 \circ d + d^\otimes \circ \delta_1 + d_{\text{ad}} \delta_1,$$

where

$$\begin{aligned} -d^\otimes \circ \delta_0 &\in \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_2), \\ d_{\text{ad}} \delta_0 &\in \text{Hom}(\wedge^2 \mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1) \oplus \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0), \\ -\delta_0 \circ d &\in \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1), \\ d^\otimes \circ \delta_1 &\in \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1), \\ d_{\text{ad}} \delta_1 &\in \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0) \end{aligned}$$

Thus (δ_0, δ_1) is a 2-cocycle if and only if the following equations hold:

$$(d \otimes 1 - 1 \otimes d) \circ \delta_0 = 0, \quad (12)$$

$$\delta_0 \circ d - (d \otimes 1 + 1 \otimes d) \circ \delta_1 = 0, \quad (13)$$

$$d_{\text{ad}} \delta_0(x, y) = 0, \quad (14)$$

$$d_{\text{ad}} \delta_0(x, h) + d_{\text{ad}} \delta_1(x, h) = 0. \quad (15)$$

3 Strict Lie 2-bialgebras

A **strict Lie 2-bialgebra** is a Lie 2-bialgebra with $c_3 = l_3 = 0$. In this section, we study it in a more classical setting. Vector spaces in this section are without gradings.

3.1 Matched pairs

We first consider how to define a strict Lie 2-algebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ of the underlying complexes of vector spaces of two strict Lie 2-algebras \mathfrak{g} and \mathfrak{g}' such that they are strict sub-Lie 2-algebras.

Theorem 3.1. *Let \mathfrak{g} and \mathfrak{g}' be two strict Lie 2-algebras, $\mu = (\mu_0, \mu_1) : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}')$, and $\mu' : \mathfrak{g}' \longrightarrow \text{End}(\mathfrak{g})$ be representations of \mathfrak{g} and \mathfrak{g}' on \mathfrak{g}' and \mathfrak{g} respectively satisfying the following compatibility conditions:*

$$\mu'_0(x')[x, y] = [x, \mu'_0(x')y] + [\mu'_0(x')x, y] + \mu'_0(\mu_0(y)x')x - \mu'_0(\mu_0(x)x')y; \quad (16)$$

$$\mu_0(x)[x', y']' = [x', \mu_0(x)y']' + [\mu_0(x)x', y']' + \mu_0(\mu'_0(y')x)x' - \mu_0(\mu'_0(x')x)y'; \quad (17)$$

$$\mu'_1(h')[x, y] = [x, \mu'_1(h')y] + [\mu'_1(h')x, y] + \mu'_1(\mu_0(y)h')x - \mu'_1(\mu_0(x)h')y; \quad (18)$$

$$\mu_1(h)[x', y']' = [x', \mu_1(h)y']' + [\mu_1(h)x', y']' + \mu_1(\mu'_0(y')h)x' - \mu_1(\mu'_0(x')h)y'; \quad (19)$$

$$\mu'_0(x')[x, h] = [x, \mu'_0(x')h] + [\mu'_0(x')x, h] + \mu'_1(\mu_1(h)x')x - \mu'_0(\mu_0(x)x')h; \quad (20)$$

$$\mu_0(x)[x', h']' = [x', \mu_0(x)h']' + [\mu_0(x)x', h']' + \mu_1(\mu'_1(h')x)x' - \mu_0(\mu'_0(x')x)h', \quad (21)$$

where $x, y \in \mathfrak{g}_0, x', y' \in \mathfrak{g}'_0, h \in \mathfrak{g}_{-1}, h' \in \mathfrak{g}'_{-1}$. Then there exists a strict Lie 2-algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}'_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}'_{-1}, d \oplus d', [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'})$, where $[\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'}$ is given by

$$[x + x', y + y']_{\mathfrak{g} \oplus \mathfrak{g}'} = [x, y] + \mu_0(x)(y') - \mu'_0(y')x + \mu'_0(x')y - \mu_0(y)x' + [x', y']'; \quad (22)$$

$$[x + x', h + h']_{\mathfrak{g} \oplus \mathfrak{g}'} = [x, h] + \mu_0(x)(h') - \mu'_1(h')(x) - \mu_1(h)x' + \mu'_0(x')(h) + [x', h']'. \quad (23)$$

Conversely, given a strict Lie 2-algebra $(\mathfrak{g}_0 \oplus \mathfrak{g}'_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}'_{-1}, d \oplus d', [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'})$ in which both \mathfrak{g} and \mathfrak{g}' with restricted brackets are strict sub-Lie 2-algebras, there exist representations up to homotopy $\mu : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}')$ and $\mu' : \mathfrak{g}' \longrightarrow \text{End}(\mathfrak{g})$ satisfying Eqs. (16)-(21) such that the bracket $[\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'}$ is given by Eqs. (22) and (23).

Proof. We need to consider the Jacobi identity

$$J(\alpha, \beta, \gamma) = [[\alpha, \beta]_{\mathfrak{g} \oplus \mathfrak{g}'}, \gamma]_{\mathfrak{g} \oplus \mathfrak{g}'} + c.p. = 0$$

for the following six cases

$$\begin{aligned} (\alpha, \beta, \gamma) \in \mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}'_0, \quad & (\alpha, \beta, \gamma) \in \mathfrak{g}'_0 \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}_0, \quad & (\alpha, \beta, \gamma) \in \mathfrak{g}_0 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}'_{-1}, \\ (\alpha, \beta, \gamma) \in \mathfrak{g}'_0 \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}_{-1}, \quad & (\alpha, \beta, \gamma) \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}'_0, \quad & (\alpha, \beta, \gamma) \in \mathfrak{g}'_0 \oplus \mathfrak{g}'_{-1} \oplus \mathfrak{g}_0, \end{aligned}$$

where “*c.p.*” means cyclic permutation. Let $x, y \in \mathfrak{g}_0$ and $x' \in \mathfrak{g}'_0$, we have

$$\begin{aligned} [[x, y]_{\mathfrak{g} \oplus \mathfrak{g}'}, x']_{\mathfrak{g} \oplus \mathfrak{g}'} &= \mu_0([x, y])x' - \mu'_0(x')[x, y]; \\ [[y, x']_{\mathfrak{g} \oplus \mathfrak{g}'}, x]_{\mathfrak{g} \oplus \mathfrak{g}'} &= [\mu_0(y)x' - \mu'_0(x')y, x]_{\mathfrak{g} \oplus \mathfrak{g}'} \\ &= [x, \mu'_0(x')y] - \mu_0(x)\mu_0(y)x' + \mu'_0(\mu_0(y)x')x; \\ [[x', x]_{\mathfrak{g} \oplus \mathfrak{g}'}, y]_{\mathfrak{g} \oplus \mathfrak{g}'} &= [\mu'_0(x')x - \mu_0(x)x', y]_{\mathfrak{g} \oplus \mathfrak{g}'} \\ &= [\mu'_0(x')x, y] + \mu_0(y)\mu_0(x)x' - \mu'_0(\mu_0(x)x')y. \end{aligned}$$

By the fact that μ is a strict homomorphism, $J(x, y, x') = 0$ if and only if Eq. (16) holds. By a similar proof for the other five cases, we obtain Eqs. (17)-(21) respectively.

Next we show that the linear map $d + d'$ is a graded derivation with respect to $[\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'}$. In fact, we only need to show:

$$\begin{aligned} (d + d')([x, h']_{\mathfrak{g} \oplus \mathfrak{g}'}) &= [x, (d + d')h']_{\mathfrak{g} \oplus \mathfrak{g}'}, \\ (d + d')([x', h]_{\mathfrak{g} \oplus \mathfrak{g}'}) &= [x', (d + d')h]_{\mathfrak{g} \oplus \mathfrak{g}'}, \\ [(d + d')h, h']_{\mathfrak{g} \oplus \mathfrak{g}'} &= [h, (d + d')h']_{\mathfrak{g} \oplus \mathfrak{g}'}. \end{aligned}$$

It is not hard to deduce that

$$\begin{aligned} (d + d')[x, h']_{\mathfrak{g} \oplus \mathfrak{g}'} &= (d + d')(\mu_0(x)h' - \mu'_1(h')x) = -d(\mu'_1(h')(x)) + d'(\mu_0(x)h'); \\ [x, (d + d')h']_{\mathfrak{g} \oplus \mathfrak{g}'} &= [x, d'h']_{\mathfrak{g} \oplus \mathfrak{g}'} = \mu_0(x)(d'h') - \mu'_0(d'h')(x). \end{aligned}$$

So $(d + d')[x, h']_{\mathfrak{g} \oplus \mathfrak{g}'} = [x, (d + d')h']$ if and only if

$$d(\mu'_1(h')(x)) = \mu'_0(d'h')(x), \quad d'(\mu_0(x)h') = \mu_0(x)(d'h'),$$

which follows from the fact that

$$\mu'_0 \circ d' = d \circ \mu'_1, \quad \mu_0(x) \circ d' = d' \circ \mu_0(x).$$

The other two equalities can be proved similarly, we omit the details.

Therefore if μ and μ' are strict representations satisfying the compatibility conditions (16)-(21), then $(\mathfrak{g}_0 \oplus \mathfrak{g}'_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}'_{-1}, d \oplus d', [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'})$ is a strict Lie 2-algebra, where the bracket operations $[\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'}$ are given by Eqs. (22)-(23).

Conversely, for any Lie 2-algebra structure $(d_{\mathfrak{g} \oplus \mathfrak{g}'}, [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}'})$ on the direct sum complex $\mathfrak{g} \oplus \mathfrak{g}'$, such that \mathfrak{g} and \mathfrak{g}' are sub-Lie 2-algebras, since $d_{\mathfrak{g} \oplus \mathfrak{g}'}|_{\mathfrak{g}} = d$ and $d_{\mathfrak{g} \oplus \mathfrak{g}'}|_{\mathfrak{g}'} = d'$, we have $d_{\mathfrak{g} \oplus \mathfrak{g}'} = d \oplus d'$. Furthermore, there exist linear maps $\mu_0, \mu_1, \mu'_0, \mu'_1$ defined by

$$\begin{aligned} [x, x']_{\mathfrak{g} \oplus \mathfrak{g}'} &= \mu_0(x)x' - \mu'_0(x')x, \\ [x, h']_{\mathfrak{g} \oplus \mathfrak{g}'} &= \mu_0(x)h' - \mu'_1(h')x, \\ [x', h]_{\mathfrak{g} \oplus \mathfrak{g}'} &= \mu'_0(x')h - \mu_1(h)x', \end{aligned}$$

where $x \in \mathfrak{g}_0, x' \in \mathfrak{g}'_0, h \in \mathfrak{g}_{-1}, h' \in \mathfrak{g}'_{-1}$. It is straightforward to check that $\mu = (\mu_0, \mu_1)$ and $\mu' = (\mu'_0, \mu'_1)$ are representations of \mathfrak{g} and \mathfrak{g}' on \mathfrak{g}' and \mathfrak{g} respectively, and satisfy the conditions (16)-(21). ■

Definition 3.2. Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ and $\mathfrak{g}' = (\mathfrak{g}'_0, \mathfrak{g}'_{-1}, d', [\cdot, \cdot]')$ be two strict Lie 2-algebras. Suppose that $\mu = (\mu_0, \mu_1) : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}')$ and $\mu' : \mathfrak{g}' \longrightarrow \text{End}(\mathfrak{g})$ are representations of \mathfrak{g} and \mathfrak{g}' on \mathfrak{g}' and \mathfrak{g} respectively. We call them a **matched pair** and denote by $(\mathfrak{g}, \mathfrak{g}'; \mu, \mu')$ if the compatibility conditions (16)-(21) are satisfied.

3.2 Manin triple

Let us recall from Definition 2.8 that a *Manin triple of strict Lie 2-algebras*, which we denote by $(\mathfrak{k}; \mathfrak{g}, \mathfrak{g}')$, consists of

- three strict Lie 2-algebras $\mathfrak{k}, \mathfrak{g}, \mathfrak{g}'$, where \mathfrak{g} and \mathfrak{g}' are sub-Lie 2-algebras of \mathfrak{k} , and as the direct sum of complexes of vector spaces, we have

$$\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}'. \quad (24)$$

- a degree 1 nondegenerate invariant symmetric bilinear form $S_{\mathfrak{k}}$ on \mathfrak{k} , such that \mathfrak{g} and \mathfrak{g}' are isotropic.

Let us explain the last item more explicitly: given a strict Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ and a bilinear form S on $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, for any $\alpha, \beta, \gamma \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, $x, y \in \mathfrak{g}_0$ and $h, k \in \mathfrak{g}_{-1}$,

- S is degree 1 symmetric bilinear if

$$S(x, y) = S(h, k) = 0, \quad S(x, h) = S(h, x);$$

- S is **nondegenerate** if

$$S(\alpha, \beta) = 0 \text{ or } S(\beta, \alpha) = 0, \forall \beta \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \iff \alpha = 0;$$

- S is **invariant** if

$$S(d\alpha, \beta) = S(\alpha, d\beta), \quad S([\alpha, \beta], \gamma) = S(\alpha, [\beta, \gamma]). \quad (25)$$

A *homomorphism* between two Manin triples $(\mathfrak{k}_1, \mathfrak{g}, \mathfrak{g}')$ and $(\mathfrak{k}_2, \mathfrak{h}, \mathfrak{h}')$ is a homomorphism $\phi : \mathfrak{k}_1 \rightarrow \mathfrak{k}_2$ of strict Lie 2-algebras satisfying

$$\phi(\mathfrak{g}) \subset \mathfrak{h}, \quad \phi(\mathfrak{g}') \subset \mathfrak{h}', \quad S_{\mathfrak{k}_1}(\alpha, \beta) = S_{\mathfrak{k}_2}(\phi(\alpha), \phi(\beta)), \quad \forall \alpha, \beta \in \mathfrak{k}. \quad (26)$$

Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ be a strict Lie 2-algebra and suppose that there is also a strict Lie 2-algebra structure on $\mathfrak{g}^* = (\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*, d^*, [\cdot, \cdot]^*)$. On the direct sum of complexes, $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \xrightarrow{d+d^*} \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^*$, there is a natural degree 1 nondegenerate symmetric bilinear form S , which is given by

$$S(x + h + x^* + h^*, y + k + y^* + k^*) = \langle x, y^* \rangle + \langle h, k^* \rangle + \langle x^*, y \rangle + \langle h^*, k \rangle, \quad (27)$$

where $x, y \in \mathfrak{g}_0, h, k \in \mathfrak{g}_{-1}, x^*, y^* \in \mathfrak{g}_0^*, h^*, k^* \in \mathfrak{g}_{-1}^*$. We call (27) the **standard bilinear form** on $\mathfrak{g} \oplus \mathfrak{g}^*$. We can introduce a bracket operation $[\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}^*}$, such that S is invariant, as follows

$$\begin{cases} [x + h^*, y + k^*]_{\mathfrak{g} \oplus \mathfrak{g}^*} \\ = [x, y] + [h^*, k^*]^* + \text{ad}_0^*(x)(k^*) - \text{ad}_{0*}^*(k^*)(x) + \text{ad}_{0*}^*(h^*)(y) - \text{ad}_0^*(y)(h^*), \\ [x + h^*, k + y^*]_{\mathfrak{g} \oplus \mathfrak{g}^*} \\ = [x, k] + [h^*, y^*] + \text{ad}_0^*(x)(y^*) - \text{ad}_{1*}^*(y^*)(x) + \text{ad}_{0*}^*(h^*)(k) - \text{ad}_1^*(h)(k^*). \end{cases} \quad (28)$$

We call this the **standard bracket operation** on $\mathfrak{g} \oplus \mathfrak{g}^*$, where $\text{ad}_*^* = (\text{ad}_{0*}^*, \text{ad}_{1*}^*)$ is the coadjoint representation of \mathfrak{g}^* on \mathfrak{g} . If $\mathfrak{g} \oplus \mathfrak{g}^* = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}^*, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^*, d + d^*, [\cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{g}^*})$ is a strict Lie 2-algebra (in this case, \mathfrak{g} and \mathfrak{g}^* are sub-Lie 2-algebras naturally), then we obtain a Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$ with respect to the standard bilinear form (27), which we call the **standard Manin triple**.

Proposition 3.3. *Any Manin triple of strict Lie 2-algebras $(\mathfrak{k}; \mathfrak{g}, \mathfrak{g}')$ with respect to a degree 1 nondegenerate invariant symmetric bilinear form $S_{\mathfrak{k}}$ is isomorphic to a standard Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$.*

Proof. Since \mathfrak{g} and \mathfrak{g}' are isotropic and the bilinear form $S_{\mathfrak{k}}$ is nondegenerate, \mathfrak{g}'_{-1} can be considered as \mathfrak{g}_0^* and \mathfrak{g}'_0 can be considered as \mathfrak{g}_{-1}^* . Furthermore, $S_{\mathfrak{k}}$ is transferred to the standard bilinear form (27). Since $S_{\mathfrak{k}}$ is invariant, we have

$$S_{\mathfrak{k}}(dh, x^*) = S_{\mathfrak{k}}(h, d'x^*),$$

which implies that

$$\langle dh, x^* \rangle = \langle h, d'x^* \rangle.$$

Thus we have

$$d' = d^*.$$

Also by the fact that $S_{\mathfrak{k}}$ is invariant, we have

$$\begin{aligned} \langle [k, x], h^* \rangle &= \langle k, [x, h^*]_{\mathfrak{k}} \rangle, \\ \langle x, [h^*, y^*]^* \rangle &= \langle y^*, [x, h^*]_{\mathfrak{k}} \rangle, \end{aligned}$$

which implies that

$$[x, h^*]_{\mathfrak{k}} = \text{ad}_0^*(x)(h^*) - \text{ad}_{0*}^*(h^*)(x).$$

Similarly, we have

$$\begin{aligned} [x, y^*]_{\mathfrak{k}} &= \text{ad}_0^*(x)(y^*) - \text{ad}_{1*}^*(y^*)(x), \\ [k, h^*]_{\mathfrak{k}} &= \text{ad}_1^*(k)(h^*) - \text{ad}_{0*}^*(h^*)(k). \end{aligned}$$

Therefore, the bracket operation on $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by Eq. (28). Thus, it is isomorphic to the standard Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$. ■

Proposition 3.4. *Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ and $\mathfrak{g}^* = (\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*, d^*, [\cdot, \cdot]^*)$ be two strict Lie 2-algebras. Then $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$ is a standard Manin triple if and only if $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair of strict Lie 2-algebras.*

Proof. It is straightforward to see that the standard bilinear form (27) is invariant under the standard bracket operation (28). Furthermore, it is also not hard to deduce that $d \oplus d^*$ is a graded derivation with respect to the standard bracket operation (28). Thus $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$ is a standard Manin triple if and only if the standard bracket operation (28) satisfies the Jacobi identity. By definition, it is equivalent to the fact that $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair. ■

3.3 Equivalence of Manin triples, matched pairs and Lie 2-bialgebras

For any linear map $\delta_1 : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$ and $\delta_0 : \mathfrak{g}_0 \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}$, define $[\cdot, \cdot]^* : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \rightarrow \mathfrak{g}_{-1}^*$, $[\cdot, \cdot]^* : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0^* \rightarrow \mathfrak{g}_0^*$ and $[\cdot, \cdot]^* : \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}^* \rightarrow \mathfrak{g}_0^*$ by

$$\langle [h^*, k^*]^*, l \rangle \triangleq \langle h^* \otimes k^*, \delta_1(l) \rangle, \quad (29)$$

$$\langle [h^*, x^*]^*, y \rangle \triangleq \langle h^* \otimes x^*, \delta_0(y) \rangle, \quad (30)$$

$$\langle [x^*, h^*]^*, y \rangle \triangleq \langle x^* \otimes h^*, \delta_0(y) \rangle. \quad (31)$$

Remark 3.5. We need to emphasize here that the above pairing $\langle \cdot, \cdot \rangle$ is the usual pairing between vector spaces, which is different from the big bracket $\langle\langle \cdot, \cdot \rangle\rangle$.

For a strict Lie 2-bialgebra $c_1 = l_1$ and $c_3 = 0$, the only non-trivial cobracket is c_2 , which corresponds to $[\cdot, \cdot]^*$, and thus to δ_0 and δ_1 . However, $c_2 \neq \delta_0 + \delta_1$. On the other hand, we do not have to make a degree -2 shift and do not have to consider the whole symmetric algebra to ensure that all the c_i 's stay in the same space as in the nonstrict case. This difference produces a slightly different cohomological explanation of the compatibility relation in Theorem 3.7.

Lemma 3.6. *Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ be a strict Lie 2-algebra, and linear maps $\delta_1 : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$ and $\delta_0 : \mathfrak{g}_0 \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}$ together with d^* define a strict Lie 2-algebra structure on \mathfrak{g}^* . Then the corresponding cobracket $c_2 \in \mathfrak{g}^*[-1] \otimes \text{Sym}^2(\mathfrak{g}[-2])$ is a 4-cocycle of \mathfrak{g} with coefficients in $\text{Sym}(\mathfrak{g}[-2])$ if and only if $\delta_0 + \delta_1$ is a 2-cocycle of \mathfrak{g} with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.*

Proof. If $c_2 = \sum l^*fg + y^*zk$ then $\delta_0 = \sum y^* \otimes (z \otimes k - k \otimes z)$ and $\delta_1 = \sum l^* \otimes (f \otimes g - g \otimes f)$. Then a routine calculation shows that $\tilde{D}_{ad}c_2 = 0$ (see Eq. (7)) if and only if $D(\delta_0 + \delta_1) = 0$ (see Eq. (11)). ■

Therefore, a strict Lie 2-bialgebra consists of a strict Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ and a 2-cocycle (δ_0, δ_1) , where $\delta_0 \in \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1})$ and $\delta_1 \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1})$ defining a semidirect product Lie algebra structure $[\cdot, \cdot]^*$ on $\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^*$ via Eqs. (29), (30) and (31). We denote a strict Lie 2-bialgebra by $(\mathfrak{g}; \delta_0, \delta_1)$.

The following theorem can be viewed as a strict version of Theorem 1.3. However, for the purpose of some potential generalization in the non-symmetric setting, our cobrackets takes value in the tensor algebra instead of the symmetric algebra as before. We provide an explicit proof in this new setting.

Theorem 3.7. *Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ be a strict Lie 2-algebra. Suppose that linear maps $\delta_1 : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$ and $\delta_0 : \mathfrak{g}_0 \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}$ define a semidirect product Lie algebra structure $[\cdot, \cdot]^*$ on $\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^*$ via Eqs. (29), (30) and (31). Then $\mathfrak{g}^* = (\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*, d^*, [\cdot, \cdot]^*)$ is a strict Lie 2-algebra such that $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair of strict Lie 2-algebras if and only if (δ_0, δ_1) is a 2-cocycle of \mathfrak{g} with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.*

Proof. Recall that (δ_0, δ_1) is a 2-cocycle if and only if Eqs. (12), (13), (14), and (15) hold. To show that $\mathfrak{g}^* = (\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*, d^*, [\cdot, \cdot]^*)$ is a strict Lie 2-algebra, we only need to show that d^* is a graded derivation (the 4th and 5th equation in Eq. (2)). By a computation, we have

$$\begin{aligned} \langle [d^*x^*, y^*]^*, z \rangle &= \langle \delta_0(z), d^*x^* \otimes y^* \rangle \\ &= \langle \delta_0(z), (d^* \otimes 1)x^* \otimes y^* \rangle = \langle (d \otimes 1)\delta_0(z), x^* \otimes y^* \rangle, \\ \langle [x^*, d^*y^*]^*, z \rangle &= \langle \delta_0(z), x^* \otimes d^*y^* \rangle \\ &= \langle \delta_0(z), (1 \otimes d^*)x^* \otimes y^* \rangle = \langle (1 \otimes d)\delta_0(z), x^* \otimes y^* \rangle. \end{aligned}$$

Thus $[d^*x^*, y^*]^* = [x^*, d^*y^*]^*$ if and only if $(d \otimes 1)\delta_0(z) = (1 \otimes d)\delta_0(z)$, which is exactly Eq. (12). Furthermore, we have

$$\begin{aligned} \langle d^*[h^*, x^*]^*, k \rangle &= \langle [h^*, x^*]^*, dk \rangle = \langle \delta_0(dk), h^* \otimes x^* \rangle, \\ \langle [h^*, d^*x^*]^*, k \rangle &= \langle \delta_1(k), h^* \otimes d^*x^* \rangle \\ &= \langle \delta_1(k), (1 \otimes d^*)h^* \otimes x^* \rangle = \langle (1 \otimes d)\delta_1(k), h^* \otimes x^* \rangle. \end{aligned}$$

Thus if $d^*[h^*, x^*]^* = [h^*, d^*x^*]^*$, we have

$$\langle \delta_0(dk), h^* \otimes x^* \rangle = \langle (1 \otimes d)\delta_1(k), h^* \otimes x^* \rangle.$$

Similarly, if we consider $d^*[x^*, h^*]^* = [d^*x^*, h^*]^*$, then we obtain that

$$\langle \delta_0(dk), x^* \otimes h^* \rangle = \langle (d \otimes 1)\delta_1(k), x^* \otimes h^* \rangle.$$

Thus, the 5-th equation in Eq. (2) holds if and only if

$$\delta_0(dk) = (d \otimes 1)\delta_1(k) + (1 \otimes d)\delta_1(k),$$

which is exactly Eq. (13).

Next we consider the conditions such that $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair of strict Lie 2-algebras, i.e., consider the compatibility conditions (16)-(21). By a computation, we have

$$\begin{aligned} \langle \text{ad}_{0*}^*(h^*)[x, y], z^* \rangle &= \langle [x, y], -[h^*, z^*]^* \rangle = \langle \delta_0([x, y]), -h^* \otimes z^* \rangle; \\ \langle [x, \text{ad}_{0*}^*(h^*)y], z^* \rangle &= \langle \text{ad}_{0*}^*(h^*)y, -\text{ad}_0^*(x)(z^*) \rangle = \langle y, [h^*, \text{ad}_0^*(x)z^*]^* \rangle \\ &= \langle \delta_0(y), h^* \otimes \text{ad}_0^*(x)z^* \rangle = \langle -(1 \otimes \text{ad}_0(x))\delta_0(y), h^* \otimes z^* \rangle; \\ \langle [\text{ad}_{0*}^*(h^*)x, y], z^* \rangle &= \langle (1 \otimes \text{ad}_0(y))\delta_0(x), h^* \otimes z^* \rangle; \\ \langle \text{ad}_{0*}^*(\text{ad}_0^*(y)h^*)x, z^* \rangle &= \langle x, -[\text{ad}_0^*(y)h^*, z^*]^* \rangle = \langle \delta_0(x), -\text{ad}_0^*(y)h^* \otimes z^* \rangle \\ &= \langle (\text{ad}_0(y) \otimes 1)\delta_0(x), h^* \otimes z^* \rangle; \\ \langle -\text{ad}_{0*}^*(\text{ad}_0^*(x)h^*)y, z^* \rangle &= \langle (\text{ad}_0(x) \otimes 1)\delta_0(y), -h^* \otimes z^* \rangle. \end{aligned}$$

Therefore, Eq. (16) holds if and only if

$$\delta_0([x, y]) = (1 \otimes \text{ad}_0(x) + \text{ad}_0(x) \otimes 1)\delta_0(y) - (1 \otimes \text{ad}_0(y) + \text{ad}_0(y) \otimes 1)\delta_0(x), \quad (32)$$

which is exactly

$$d_{\text{ad}}\delta_0(x, y) = 0.$$

Similarly, we have

$$\begin{aligned} \langle \text{ad}_0^*(x)[h^*, k^*]^*, k \rangle &= \langle -[x, k], [h^*, k^*]^* \rangle = \langle \delta_1([x, k]), -h^* \otimes k^* \rangle; \\ \langle [h^*, \text{ad}_0^*(x)k^*]^*, k \rangle &= \langle \delta_1(k), h^* \otimes \text{ad}_0^*(x)k^* \rangle = \langle (1 \otimes \text{ad}_0(x))\delta_1(k), -h^* \otimes k^* \rangle; \\ \langle [\text{ad}_0^*(x)h^*, k^*]^*, k \rangle &= \langle (\text{ad}_0(x) \otimes 1)\delta_1(k), -h^* \otimes k^* \rangle; \\ \langle -\text{ad}_0^*(\text{ad}_{0*}^*(h^*)x)k^*, k \rangle &= \langle [\text{ad}_{0*}^*(h^*)x, k], k^* \rangle = \langle \text{ad}_{0*}^*(h^*)x, \text{ad}_1^*(k)k^* \rangle = \langle x, -[h^*, \text{ad}_1^*(k)k^*]^* \rangle \\ &= \langle \delta_0(x), -h^* \otimes \text{ad}_1^*(k)k^* \rangle = \langle (1 \otimes \text{ad}_1(k))\delta_0(x), h^* \otimes k^* \rangle. \\ \langle \text{ad}_0^*(\text{ad}_{0*}^*(k^*)x)h^*, k \rangle &= \langle x, [k^*, \text{ad}_1^*(k)h^*]^* \rangle = \langle (\text{ad}_1(k) \otimes 1)\delta_0(x), h^* \otimes k^* \rangle. \end{aligned}$$

Therefore, Eq. (17) holds if and only if

$$\delta_1([x, k]) = (1 \otimes \text{ad}_0(x) + \text{ad}_0(x) \otimes 1)\delta_1(k) - (1 \otimes \text{ad}_1(k) + \text{ad}_1(k) \otimes 1)\delta_0(x), \quad (33)$$

which means that

$$d_{\text{ad}}\delta_0(x, k) + d_{\text{ad}}\delta_1(x, k) = 0.$$

Similarly, one can deduce that Eqs. (18) and (21) hold if and only if Eq. (32) holds, and Eqs. (19) and (20) hold if and only if Eq. (33) holds. The proof is completed. ■

To summarize, we have:

Corollary 3.8. *Let \mathfrak{g} be a strict Lie 2-algebra. Then the following conditions are equivalent:*

1. $(\mathfrak{g}; \delta_0, \delta_1)$ is a strict Lie 2-bialgebra;
2. $(\mathfrak{g}^*, [\cdot, \cdot]^*)$ with $[\cdot, \cdot]^*$ given by Eqs. (29) -(31) is a strict Lie 2-algebra and $(\mathfrak{g}, \mathfrak{g}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair of strict Lie 2-algebras;
3. $(\mathfrak{g}^*, [\cdot, \cdot]^*)$ with $[\cdot, \cdot]^*$ given by Eqs. (29) -(31) is a strict Lie 2-algebra and $(\mathfrak{g} \oplus \mathfrak{g}^*; \mathfrak{g}, \mathfrak{g}^*)$ is a standard Manin triple with respect to the standard bilinear form (27).

Remark 3.9 (Lie bialgebras as strict Lie 2-bialgebras). For an arbitrary Lie algebra \mathfrak{h} , it is obvious that $\mathfrak{h} \xrightarrow{\text{Id}} \mathfrak{h}$ is a strict Lie 2-algebra. The degree 0 part \mathfrak{h} acts on the degree -1 \mathfrak{h} via the Lie bracket. If $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra, it is easy to see that $(\mathfrak{h} \xrightarrow{\text{Id}} \mathfrak{h}, \mathfrak{h}^* \xrightarrow{\text{Id}} \mathfrak{h}^*; \text{ad}^*, \text{ad}_*^*)$ is a matched pair of strict Lie 2-algebras, thus $(\mathfrak{h} \xrightarrow{\text{Id}} \mathfrak{h}; \delta_0, \delta_1)$ is a strict Lie 2-bialgebra, where δ_0, δ_1 are both given by the Lie algebra structures on \mathfrak{h}^* . Thus the above provides a way to embed the category of Lie bialgebras in the category of Lie 2-bialgebras.

3.4 Coboundary strict Lie 2-bialgebras and classical Yang-Baxter equations

In this section, we consider coboundary strict Lie 2-bialgebras, i.e., (δ_0, δ_1) is an exact 2-cocycle. For any 1-cochain $(r, \phi) \in (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_1 \oplus \text{Hom}(\mathfrak{g}_0, (\mathfrak{g}[-1] \otimes \mathfrak{g}[-1])_0)$, we have (for notations, see Sect. 2.4):

$$\begin{aligned} D(r, \phi) &= \widehat{d}^{\otimes} r + d_{\text{ad}} r + \widehat{d}^{\otimes} \phi + \widehat{d} \phi + d_{\text{ad}} \phi \\ &= d^{\otimes} r + d_{\text{ad}} r - d^{\otimes} \circ \phi - \phi \circ d + d_{\text{ad}} \phi, \end{aligned}$$

where

$$\begin{aligned} d^{\otimes} r &= (d \otimes 1 - 1 \otimes d) r \in (\mathfrak{g} \otimes \mathfrak{g})_0, \\ d_{\text{ad}} r &\in \text{Hom}(\mathfrak{g}_0, (\mathfrak{g} \otimes \mathfrak{g})_{-1}) \oplus \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g} \otimes \mathfrak{g})_{-2}), \\ d^{\otimes} \circ \phi + \phi \circ d &= (d \otimes 1 + 1 \otimes d) \circ \phi + \phi \circ d \in \text{Hom}(\mathfrak{g}_0, (\mathfrak{g} \otimes \mathfrak{g})_{-1}) \oplus \text{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g} \otimes \mathfrak{g})_{-2}), \\ d_{\text{ad}} \phi &\in \text{Hom}(\wedge^2 \mathfrak{g}_0, (\mathfrak{g} \otimes \mathfrak{g})_{-2}). \end{aligned}$$

Therefore, $(\delta_0, \delta_1) = D(r, \phi)$ for some 1-cochain (r, ϕ) , we must have

$$(d \otimes 1 - 1 \otimes d) r = 0, \quad (34)$$

$$d_{\text{ad}} \phi = 0,$$

$$\delta_0(x) = d_{\text{ad}} r(x) + \widehat{d}^{\otimes} \phi(x) = [x \otimes 1 + 1 \otimes x, r] - d^{\otimes} \circ \phi(x), \quad (35)$$

$$\delta_1(h) = d_{\text{ad}} r(h) + \widehat{d} \phi(h) = [h \otimes 1 + 1 \otimes h, r] - \phi(dh). \quad (36)$$

Proposition 3.10. Let δ_0 and δ_1 be given by Eqs. (35) and (36) for some (r, ϕ) satisfying Eq. (34). If

$$[\alpha \otimes 1 + 1 \otimes \alpha, r + \sigma(r)] = 0, \quad \forall \alpha \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1},$$

where $\sigma : (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \otimes (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \rightarrow (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}) \otimes (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})$ is the exchanging operator, i.e., $\sigma(\alpha \otimes \beta) = \beta \otimes \alpha$, for any $\alpha, \beta \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, and

$$\phi^*(h^* \otimes k^*) + \phi^*(k^* \otimes h^*) = 0,$$

then the bracket operation $[\cdot, \cdot]^*$ defined by Eqs. (29), (30) and (31) are skew-symmetric. Under this assumption, we have

$$[h^*, k^*]^* = [h^*, k^*]_r - d^* \phi^*(h^* \otimes k^*), \quad (37)$$

$$[h^*, x^*]^* = [h^*, x^*]_r - \phi^*(h^* \otimes d^* x^*), \quad (38)$$

where $[h^*, k^*]_r$ and $[h^*, x^*]_r$ are given by

$$\begin{aligned} [h^*, k^*]_r &\triangleq \text{ad}_{r(h^*)} k^* - \text{ad}_{r(k^*)} h^*, \\ [h^*, x^*]_r &\triangleq \text{ad}_{r(h^*)} x^* - \text{ad}_{r(x^*)} h^*. \end{aligned}$$

Furthermore, $[\cdot, \cdot]^* : \wedge^2 \mathfrak{g}_{-1}^* \longrightarrow \mathfrak{g}_{-1}^*$ satisfies the Jacobi identity if and only if

$$[[h^*, k^*]_r, l^*]_r - d^* \phi^*([h^*, k^*]_r, l^*) - [d^* \phi^*(h^*, k^*), l^*]_r + d^* \phi^*(d^* \phi^*(h^*, k^*), l^*) + c.p. = 0, \quad (39)$$

and $[\cdot, \cdot]^* : \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_0^*$ satisfies the Jacobi identity if and only if

$$\begin{aligned} &[[h^*, k^*]_r, x^*]_r + c.p. \\ &- \phi^*([h^*, k^*]_r, d^* x^*) - \phi^*(d^*[k^*, x^*]_r, h^*) - \phi^*(d^*[x^*, h^*]_r, k^*) \\ &- [d^* \phi^*(h^*, k^*), x^*]_r - [\phi^*(k^*, d^* x^*), h^*]_r - [\phi^*(d^* x^*, h^*), k^*]_r \\ &+ \phi^*(d^* \phi^*(h^*, k^*), d^* x^*) + \phi^*(d^* \phi^*(k^*, d^* x^*), h^*) + \phi^*(d^* \phi^*(d^* x^*, h^*), k^*) = 0. \end{aligned} \quad (40)$$

Proof. First we have

$$\begin{aligned} \langle [h^*, k^*]^*, l \rangle &= \langle \delta_1(l), h^* \otimes k^* \rangle \\ &= \langle [l \otimes 1 + 1 \otimes l, r] - \phi(dl), h^* \otimes k^* \rangle \\ &= \langle [l \otimes 1 + 1 \otimes l, r], h^* \otimes k^* \rangle - \langle dl, \phi^*(h^* \otimes k^*) \rangle. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle [k^*, h^*]^*, l \rangle &= \langle \delta_1(l), k^* \otimes h^* \rangle \\ &= \langle [l \otimes 1 + 1 \otimes l, r] - \phi(dl), k^* \otimes h^* \rangle \\ &= \langle (\text{ad}_l \otimes 1 + 1 \otimes \text{ad}_l)r, k^* \otimes h^* \rangle - \langle dl, \phi^*(k^* \otimes h^*) \rangle \\ &= -\langle r, (\text{ad}_l^* \otimes 1 + 1 \otimes \text{ad}_l^*)(k^* \otimes h^*) \rangle - \langle dl, \phi^*(k^* \otimes h^*) \rangle \\ &= -\langle r, \text{ad}_l^* k^* \otimes h^* + k^* \otimes \text{ad}_l^* h^* \rangle - \langle dl, \phi^*(k^* \otimes h^*) \rangle \\ &= -\langle \sigma(r), h^* \otimes \text{ad}_l^* k^* + \text{ad}_l^* h^* \otimes k^* \rangle - \langle dl, \phi^*(k^* \otimes h^*) \rangle \\ &= \langle [l \otimes 1 + 1 \otimes l, \sigma(r)], h^* \otimes k^* \rangle - \langle dl, \phi^*(k^* \otimes h^*) \rangle. \end{aligned}$$

Thus if $\phi^*(h^* \otimes k^*) + \phi^*(k^* \otimes h^*) = 0$ and $[l \otimes 1 + 1 \otimes l, r + \sigma(r)] = 0$ for any $l \in \mathfrak{g}_{-1}$, then we deduce that $[\cdot, \cdot]^*$ defined by Eq. (29) is skew-symmetric. Similarly, one can prove that if $\phi^*(h^* \otimes k^*) + \phi^*(k^* \otimes h^*) = 0$ and $[x \otimes 1 + 1 \otimes x, r + \sigma(r)] = 0$ for any $x \in \mathfrak{g}_0$, $[\cdot, \cdot]^*$ defined by Eqs. (30) and (31) are skew-symmetric.

By computations, we have

$$\begin{aligned} \langle [l \otimes 1 + 1 \otimes l, r], h^* \otimes k^* \rangle &= \langle r, -\text{ad}_l^*(h^*) \otimes k^* - h^* \otimes \text{ad}_l^*(k^*) \rangle \\ &= \langle r(k^*), \text{ad}_l^*(h^*) \rangle - \langle r(h^*), \text{ad}_l^*(k^*) \rangle \\ &= -\langle \text{ad}_l r(k^*), h^* \rangle + \langle \text{ad}_l r(h^*), k^* \rangle \\ &= \langle \text{ad}_{r(k^*)} l, h^* \rangle - \langle \text{ad}_{r(h^*)} l, k^* \rangle \\ &= -\langle l, \text{ad}_{r(k^*)}^* h^* \rangle + \langle l, \text{ad}_{r(h^*)}^* k^* \rangle, \end{aligned}$$

The second equality holds since $[\cdot, \cdot]^*$ is skew-symmetric. Thus $[\cdot, \cdot]^* : \wedge^2 \mathfrak{g}_{-1}^* \longrightarrow \mathfrak{g}_{-1}^*$ is given by Eq. (37). It is straightforward to deduce that it satisfies the Jacobi identity if and only if Eq. (39) holds.

Similarly, one can prove that $[\cdot, \cdot]^* : \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_0^*$ is given by Eq. (38) and it satisfies the Jacobi identity if and only if Eq. (40) holds. ■

We require $d_{\text{ad}}\phi = 0$, in particular, we choose $\phi = d_{\text{ad}}\mathfrak{r}$ for some $\mathfrak{r} \in \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$.

Proposition 3.11. *If $\phi = d_{\text{ad}}\mathfrak{r}$ for some $\mathfrak{r} \in \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$, then we have*

$$\delta_0(x) = [x \otimes 1 + 1 \otimes x, r - d^{\otimes}\mathfrak{r}], \quad \delta_1(h) = [h \otimes 1 + 1 \otimes h, r - d^{\otimes}\mathfrak{r}]. \quad (41)$$

Proof. Since $D^2 = 0$, we have

$$\widehat{d^{\otimes}} \circ d_{\text{ad}}\mathfrak{r} + d_{\text{ad}} \circ \widehat{d^{\otimes}}\mathfrak{r} = 0,$$

which implies that

$$d_{\text{ad}} \circ d^{\otimes}\mathfrak{r} = d^{\otimes} \circ d_{\text{ad}}\mathfrak{r}.$$

Thus, we have

$$\delta_0(x) = d_{\text{ad}}r(x) - d_{\text{ad}}(d^{\otimes}\mathfrak{r})(x) = d_{\text{ad}}(r - d^{\otimes}\mathfrak{r})(x) = [x \otimes 1 + 1 \otimes x, r - d^{\otimes}\mathfrak{r}].$$

Also by $D^2 = 0$, we have

$$\widehat{d}(d_{\text{ad}}\mathfrak{r}) + d_{\text{ad}}(\widehat{d^{\otimes}}\mathfrak{r}) = 0,$$

which implies that

$$d_{\text{ad}}\mathfrak{r}(dh) = d_{\text{ad}}(d^{\otimes}\mathfrak{r})(h).$$

Thus, we have

$$\delta_1(h) = d_{\text{ad}}r(h) - d_{\text{ad}}\mathfrak{r}(dh) = d_{\text{ad}}(r - d^{\otimes}\mathfrak{r})(h) = [h \otimes 1 + 1 \otimes h, r - d^{\otimes}\mathfrak{r}].$$

The proof is completed. ■

The following well known conclusion can be found in [11]:

Lemma 3.12. *Let \mathfrak{h} be a Lie algebra and $\delta : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ be a linear map. If there exists $r \in \mathfrak{h} \otimes \mathfrak{h}$ such that*

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \quad \forall x \in \mathfrak{h},$$

and then $\delta^ : \mathfrak{h}^* \otimes \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ defines a Lie algebra structure if and only if r satisfies*

- $[x \otimes 1 + 1 \otimes x, r + \sigma(r)] = 0;$
- $[x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, [r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}]] = 0,$

where the above equations make sense in the universal enveloping algebra of \mathfrak{h} and for $r = \sum_i a_i \otimes b_i$,

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i; \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \quad (42)$$

Remark 3.13. In particular, the following equation

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0 \quad (43)$$

is called classical Yang-Baxter equation (CYBE) in the Lie algebra \mathfrak{h} . The corresponding matrix to a solution r of the CYBE is called a classical r -matrix.

Theorem 3.14. *Let $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_{-1}, d, [\cdot, \cdot])$ be a strict Lie 2-algebra with two linear maps δ_0 and δ_1 given by Eq. (41) for $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0$ and $\mathfrak{r} \in \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}$. Then $(\mathfrak{g}; \delta_0, \delta_1)$ is a strict Lie 2-bialgebra if and only if the following conditions are satisfied:*

- (a) $[\alpha \otimes 1 + 1 \otimes \alpha, R + \sigma(R)] = 0,$
- (b) $[\alpha \otimes 1 \otimes 1 + 1 \otimes \alpha \otimes 1 + 1 \otimes 1 \otimes \alpha, [R_{12}, R_{13}] + [R_{13}, R_{23}] + [R_{12}, R_{23}]] = 0,$
- (c) $d^{\otimes} r = 0,$

for any $\alpha \in \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, where $R = r - d^{\otimes} \mathfrak{r} = r - (d \otimes 1 + 1 \otimes d)\mathfrak{r}$.

Proof. Since $(\delta_0, \delta_1) = D(r, d_{\text{ad}}\phi)$ is an exact cocycle, by Theorem 3.7, we only need to show that $[\cdot, \cdot]^*$ given by Eqs. (29), (30) and (31) define a semidirect product Lie algebra structure on $\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^*$. The conclusion follows from Proposition 3.11 and Lemma 3.12. ■

Inspired by this theorem, we correspondingly call the classical Yang-Baxter equations for R together with $d^{\otimes} r = 0$ the *higher classical Yang-Baxter equations* (higher CYBE) for r and \mathfrak{r} .

Remark 3.15. By Proposition 3.10, we have seen that for coboundary strict Lie 2-bialgebras, there are more general r -matrices, which are certain pairs (r, ϕ) . However, without requiring that $\phi = d_{\text{ad}}\mathfrak{r}$, it is not easy to write down the equations that they need to obey.

4 Constructions of strict Lie 2-bialgebras from left-symmetric algebras and symplectic Lie algebras

In this section, we give explicit examples of strict Lie 2-bialgebras by solving the higher CYBE. These solutions are constructed from left-symmetric algebras and symplectic Lie algebras. We consider the case where both δ_0 and δ_1 are given by $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0$, i.e.,

$$\begin{aligned} \delta_0(x) &= [x \otimes 1 + 1 \otimes x, r], \\ \delta_1(h) &= [h \otimes 1 + 1 \otimes h, r]. \end{aligned}$$

By Theorem 3.14, if r is a solution of CYBE in $\mathfrak{g}_0 \ltimes \mathfrak{g}_{-1}$ and satisfies $d^{\otimes} r = 0$, then r gives a strict Lie 2-bialgebra structure on the strict Lie 2-algebra \mathfrak{g} .

4.1 Examples from left-symmetric algebras

Definition 4.1. A **left-symmetric algebra**, A , is a vector space equipped with a bilinear product $(x, y) \rightarrow x \circ y$ satisfying that for any $x, y, z \in A$, the associator

$$(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)$$

is symmetric in x, y , i.e.,

$$(x, y, z) = (y, x, z), \text{ or equivalently, } (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

For any $x \in A$, let L_x denote the left multiplication operator, i.e., $L_x(y) = x \circ y$ for any $y \in A$. The following conclusion is known ([33, 21, 18, 10]):

Lemma 4.2. *Let A be a left-symmetric algebra. The commutator*

$$[x, y] = x \circ y - y \circ x, \quad \forall x, y \in A,$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called a compatible left-symmetric algebra structure on the Lie algebra $\mathfrak{g}(A)$. Furthermore, $L : \mathfrak{g}(A) \rightarrow \mathfrak{gl}(A)$ with $x \rightarrow L_x$ gives a representation of the Lie algebra $\mathfrak{g}(A)$, i.e., $[L_x, L_y] = L_{[x, y]}$.

Let L^* be the dual representation of the Lie algebra $\mathfrak{g}(A)$ on A^* . Then there is a semidirect product Lie algebra structure $[\cdot, \cdot]_s$ on $\mathfrak{g}(A) \oplus A^*$, which is given by

$$[x + h, y + k]_s = [x, y] + L_x^*k - L_y^*h. \quad (44)$$

We denote the corresponding Lie algebra by $\mathfrak{g}(A) \ltimes_{L^*} A^*$. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be the dual basis of A^* . We have

Theorem 4.3. [6] *Let A be a left-symmetric algebra. Then*

$$r = \sum_{i=1}^n (e_i \otimes e_i^* - e_i^* \otimes e_i) \quad (45)$$

is a solution of CYBE in $\mathfrak{g}(A) \ltimes_{L^} A^*$.*

This motivates us to construct (coboundary) strict Lie 2-bialgebras from left-symmetric algebras by constructing an explicit solution of higher CYBE: Let $\mathfrak{g}_0 = \mathfrak{g}(A)$, $\mathfrak{g}_{-1} = A^*$. Then $r \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_0$ given by Eq. (45) is a solution of CYBE in $\mathfrak{g}(A) \ltimes_{L^*} A^*$. Comparing to higher CYBE we need to take care of one more equation: $d^{\otimes} r = 0$. This leads to the following proposition, which follows by a straightforward calculation:

Proposition 4.4. *Let A be a left-symmetric algebra and $[\cdot, \cdot]_s$ given by Eq. (44). If a linear map $d : A^* \rightarrow A$ satisfies*

$$(i) \quad d[x, h]_s = [x, dh]_s, \quad [dh, k]_s = [h, dk]_s;$$

$$(ii) \quad (d \otimes 1 - 1 \otimes d)r = 0,$$

where r is given by Eq. (45), then r defines a strict Lie 2-bialgebra structure on the strict Lie 2-algebra $(\mathfrak{g}(A), A^, d, [\cdot, \cdot]_s)$.*

Example 4.5. *It is obvious that $d = 0$ satisfies conditions in Proposition 4.4, i.e., for any left-symmetric algebra A , $(\mathfrak{g}(A), A^*, 0, [\cdot, \cdot]_s)$ is a strict Lie 2-algebra and thus there is a strict Lie 2-bialgebra induced by r given by Eq. (45).*

In general, assume that

$$d(e_i^*) = \sum_{j=1}^n d_{ij} e_j, \quad i = 1, \dots, n. \quad (46)$$

Let $M(d) = (d_{ij})$ be the corresponding matrix. The following conclusion is straightforward.

Lemma 4.6. *Let A be a vector space and $d : A^* \rightarrow A$ be a linear map. Suppose that r is given by Eq. (45). Then $(d \otimes 1 - 1 \otimes d)r = 0$ if and only if $M(d)$ is skew-symmetric.*

Example 4.7. *The 1-dimensional non-trivial left-symmetric algebra is isomorphic to the field, that is, there is a basis $\{e\}$ satisfying $e \circ e = e$. In this case, it is straightforward to know that d satisfies Condition (i) in Proposition 4.4 if and only if $d = 0$.*

The classification of 2-dimensional complex left-symmetric algebras was given in [8, 9].

Example 4.8. A non-trivial left-symmetric algebra with d satisfying Condition (i) in Proposition 4.4 is isomorphic to one of the followings (we only give the non-zero products):

- A1. $e_1 \circ e_1 = e_1, e_2 \circ e_2 = e_2 : d = 0$;
- A2. $e_1 \circ e_1 = e_1 : M(d) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, a, b \in \mathbb{C}$;
- A3. $e_1 \circ e_1 = e_1, e_1 \circ e_2 = e_2 \circ e_1 = e_2 : d = 0$;
- A4. $e_1 \circ e_1 = e_2 : M(d) = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, a, b \in \mathbb{C}$;
- N1. $e_2 \circ e_1 = -e_1, e_2 \circ e_2 = ke_2, k \neq 1$ or $e_2 \circ e_2 = e_1 - e_2 : d = 0$;
- N2. $e_2 \circ e_1 = -e_1, e_2 \circ e_2 = e_2 : M(d) = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, a \in \mathbb{C}$;
- N3. $e_1 \circ e_1 = e_1, e_2 \circ e_1 = e_2 : M(d) = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, a \in \mathbb{C}$;
- N4. $e_1 \circ e_2 = le_1, e_2 \circ e_1 = (l-1)e_1, e_2 \circ e_2 = e_1 + le_2, l \neq 0, 1 : d = 0$;
- N5. $e_1 \circ e_2 = e_1, e_2 \circ e_2 = e_1 + e_2 : M(d) = \begin{pmatrix} a & -a \\ a & 0 \end{pmatrix}, a \in \mathbb{C}$;
- N6. $e_1 \circ e_1 = 2e_1, e_1 \circ e_2 = e_2, e_2 \circ e_2 = e_1 : d = 0$.

Hence, type (N3) is the only case that $M(d)$ is skew-symmetric and nonzero. Note that (N3) is associative. In addition, it is a Novikov algebra (a left-symmetric algebra with commuting right multiplication operators), which corresponds to the Poisson brackets of one-dimensional hydrodynamics [5]. Moreover, it gives the so-called ‘‘conformal current type Lie algebras’’ in terms of the Balinsky-Novikov’s affinization [27].

Example 4.9. By the classification in the above example, it is straightforward to see that \mathbb{R}^2 has the following left-symmetric algebra structure

$$e_1 \circ e_1 = e_1, \quad e_2 \circ e_1 = e_2,$$

where $\{e_1, e_2\}$ is a basis of \mathbb{R}^2 . The corresponding Lie algebra structure $(\mathfrak{g}(\mathbb{R}^2), [\cdot, \cdot])$ is

$$[e_2, e_1] = e_2 \circ e_1 - e_1 \circ e_2 = e_2. \quad (47)$$

The dual representation of $\mathfrak{g}(\mathbb{R}^2)$ on $\mathfrak{g}^*(\mathbb{R}^2)$ is given by

$$L_{e_1}^* e_1^* = -e_1^*, \quad L_{e_1}^* e_2^* = 0, \quad L_{e_2}^* e_1^* = 0, \quad L_{e_2}^* e_2^* = -e_1^*, \quad (48)$$

where $\{e_1^*, e_2^*\}$ is the dual basis. Any $M(d) = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$, $a \in \mathbb{R}$ satisfies Condition (i) in Proposition 4.4. In particular, let $a = 1$, then we have

$$d(e_1^*) = -e_2, \quad d(e_2^*) = e_1.$$

Then we obtain a strict Lie 2-algebra $(\mathfrak{g}(\mathbb{R}^2), \mathfrak{g}^*(\mathbb{R}^2), d, [\cdot, \cdot]_s)$, where $[\cdot, \cdot]_s$ is determined by Eqs. (44), (47) and (48). By Proposition 4.4, r given by (45) defines a strict Lie 2-bialgebra structure on the strict Lie 2-algebra $(\mathfrak{g}(\mathbb{R}^2), \mathfrak{g}^*(\mathbb{R}^2), d, [\cdot, \cdot]_s)$. More precisely, we have

$$\begin{aligned} \delta_0(e_1) &= [e_1 \otimes 1 + 1 \otimes e_1, r] = -r = e_1^* \otimes e_1 - e_1 \otimes e_1^* + e_2^* \otimes e_2 - e_2 \otimes e_2^*, \\ \delta_0(e_2) &= 0, \\ \delta_1(e_1^*) &= 0, \\ \delta_1(e_2^*) &= e_1^* \otimes e_2 - e_2^* \otimes e_1. \end{aligned}$$

The dual complex of $\mathfrak{g}^*(\mathbb{R}^2) \xrightarrow{d} \mathfrak{g}(\mathbb{R}^2)$ is $\mathfrak{g}^*(\mathbb{R}^2) \xrightarrow{d^*} \mathfrak{g}(\mathbb{R}^2)$. The Lie 2-algebra structure $[\cdot, \cdot]^*$ on the dual complex is given by

$$[e_1, e_2]^* = e_2, \quad [e_1, e_1^*]^* = e_1^*, \quad [e_2, e_2^*]^* = e_1^*.$$

In fact, we have

$$\begin{aligned} \langle [e_1, e_2]^*, e_2^* \rangle &= \langle \delta_1(e_2^*), e_1 \otimes e_2 \rangle = 1, \\ \langle [e_1, e_2]^*, e_1^* \rangle &= \langle \delta_1(e_1^*), e_1 \otimes e_2 \rangle = 0, \end{aligned}$$

which implies that $[e_1, e_2]^* = e_2$, the others can be obtained similarly.

4.2 Examples from symplectic Lie algebras

In the sequel, we consider the case that d is invertible. We find that it has a close relationship with symplectic Lie algebras, which leads to an unexpected construction of strict Lie 2-bialgebras from the latter.

Let $d : A^* \rightarrow A$ be an invertible linear map such that $M(d)$ is skew-symmetric. Then

$$B_d(x, y) = \langle d^{-1}(x), y \rangle \tag{49}$$

is a skew-symmetric nondegenerate bilinear form on A . Let (A, \circ) be a left-symmetric algebra. A skew-symmetric bilinear form ω on (A, \circ) is called **invariant** if ω satisfies

$$\omega(x \circ y, z) + \omega(z \circ x, y) - \omega(x \circ z, y) = 0,$$

or equivalently,

$$\omega(x \circ y, z) = \omega([x, z], y), \quad \forall x, y, z \in A.$$

Proposition 4.10. *Let (A, \circ) be a left-symmetric algebra and $d : A^* \rightarrow A$ be an invertible linear map such that $M(d)$ is skew-symmetric. Then $(\mathfrak{g}(A), A^*, d, [\cdot, \cdot]_s)$ is a strict Lie 2-algebra if and only if B_d is invariant. In this case, r given by Eq. (45) gives rise to a Lie 2-bialgebra structure.*

Proof. $(\mathfrak{g}(A), A^*, d, [\cdot, \cdot]_s)$ is a strict Lie 2-algebra if and only if d satisfies

$$d[x, h]_s = [x, dh]_s, \quad (50)$$

$$[h, dk]_s = [dh, k]_s, \quad (51)$$

for any $x \in A, h, k \in A^*$. Set $dh = y, dk = z$. Then Eq. (50) holds if and only if

$$\begin{aligned} B_d([x, y]_s, z) &= B_d([x, dh]_s, z) = B_d(d[x, h]_s, z) = \langle [x, h]_s, z \rangle = \langle h, -x \circ z \rangle \\ &= B_d(dh, -x \circ z) = -B_d(y, x \circ z) = B_d(x \circ z, y), \end{aligned}$$

which is exactly the condition that B_d is invariant. On the other hand, Eq. (51) holds if and only if

$$B_d(y, z \circ x) = \langle h, dk \circ x \rangle = \langle [h, dk]_s, x \rangle = \langle [dh, k]_s, x \rangle = -\langle k, dh \circ x \rangle = -B_d(z, y \circ x). \quad (52)$$

It can be obtained if B_d is invariant:

$$B_d(y, z \circ x) = -B_d(z \circ x, y) = -B_d([z, y], x) = B_d([y, z], x) = B_d(y \circ x, z) = -B_d(z, y \circ x).$$

Hence, the conclusion holds. ■

Let \mathfrak{h} be a Lie algebra, recall that a skew-symmetric bilinear form ω is called a 2-cocycle if ω satisfies

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \quad \forall x, y, z \in \mathfrak{h}.$$

A **symplectic Lie algebra** is a pair (\mathfrak{h}, ω) , where \mathfrak{h} is a Lie algebra and ω is a nondegenerate 2-cocycle. The following conclusion is given in [14]:

Proposition 4.11. *Let (\mathfrak{h}, ω) be a symplectic Lie algebra. Then there exists a compatible left-symmetric algebra structure “ \circ ” on \mathfrak{h} given by*

$$\omega(x \circ y, z) = -\omega(y, [x, z]), \quad \forall x, y, z \in \mathfrak{h}. \quad (53)$$

Moreover, the Lie algebra \mathfrak{h} is the sub-adjacent Lie algebra of this left-symmetric algebra.

Proposition 4.12. *If ω is an invariant skew-symmetric bilinear form on a left-symmetric algebra (A, \circ) , then ω is a 2-cocycle of the sub-adjacent Lie algebra $\mathfrak{g}(A)$; Conversely, if (\mathfrak{h}, ω) is a symplectic Lie algebra, then ω is invariant with respect to the compatible left-symmetric algebra structure given by Eq. (53).*

Proof. If ω is invariant on (A, \circ) , then we have

$$\omega(x \circ y, z) - \omega(y \circ x, z) = \omega([x, y], z) = \omega([x, z], y) - \omega([y, z], x).$$

So ω is a 2-cocycle of the sub-adjacent Lie algebra $\mathfrak{g}(A)$. The second half part is obvious. ■

This shows that a left-symmetric algebra with a nondegenerate (skew-symmetric) invariant bilinear form is equivalent to a symplectic Lie algebra.

Summarizing the study in this section, we have the following result.

Theorem 4.13. *Let A be a left-symmetric algebra and $d : A^* \rightarrow A$ be an invertible linear map such that $M(d)$ is skew-symmetric. Then the following conditions are equivalent:*

1. $(\mathfrak{g}(A), A^*, d, [\cdot, \cdot]_s)$ is a strict Lie 2-algebra;
2. The bilinear form B_d induced by d through Eq. (49) is invariant on A ;
3. The sub-adjacent Lie algebra $\mathfrak{g}(A)$ is a symplectic Lie algebra with the symplectic form B_d induced by d through Eq. (49).

Corollary 4.14. Let (\mathfrak{h}, ω) be a symplectic Lie algebra. Denote by (A, \circ) the corresponding left symmetric algebra. Then $(\mathfrak{g}(A) = \mathfrak{h}, A^*, d, [\cdot, \cdot]_s)$ is a strict Lie 2-algebra, where $[\cdot, \cdot]_s$ is given by the semidirect product Lie algebra structures on $\mathfrak{g}(A) \ltimes_{L^*} A^*$, in which the compatible left-symmetric algebra structure is given by Eq. (53), and d is given by Eq. (49). Moreover, r given by Eq. (45) defines a strict Lie 2-bialgebra structure on $(\mathfrak{g}(A), A^*, d, [\cdot, \cdot]_s)$.

Example 4.15. The study of symplectic Lie algebras is fruitful. In particular, there is a bialgebra theory of left-symmetric algebras which are equivalent to a special class of symplectic Lie algebras that are decomposed into a direct sum of two Lagrangian subalgebras [7]. In the simplest case, for any left-symmetric algebra (A, \circ) , there is a natural symplectic Lie algebra structure on $A \oplus A^*$ whose Lie algebra structure is given by $\mathfrak{g}(A) \ltimes_{L^*} A^*$ and the symplectic form is given by

$$\omega_p(x + a^*, y + b^*) = \langle a^*, y \rangle - \langle x, b^* \rangle, \quad \forall x, y \in A, a^*, b^* \in A^*. \quad (54)$$

The compatible left-symmetric algebra structure, which we denote by $\bar{\circ}$, on this symplectic Lie algebra defined by Eq. (53) is given by

$$x \bar{\circ} y = x \circ y, \quad x \bar{\circ} a^* = \text{ad}_x^* a^*, \quad a^* \bar{\circ} x = \text{ad}_x^* a^* - L_x^* a^*, \quad a^* \bar{\circ} b^* = 0, \quad (55)$$

for any $x, y \in A, a^*, b^* \in A^*$. Set $\hat{A} = A \oplus A^*$. Let $\{e_1, \dots, e_n\}$ be a basis of A and $\{e_1^*, \dots, e_n^*\}$ be the dual basis on A^* . Let $\{f_1, \dots, f_n, f_1^*, \dots, f_n^*\}$ be the corresponding dual basis on \hat{A}^* , i.e.,

$$\langle f_i, e_j \rangle = \delta_{ij}, \quad \langle f_i, e_j^* \rangle = 0, \quad \langle f_i^*, e_j \rangle = 0, \quad \langle f_i^*, e_j^* \rangle = \delta_{ij}.$$

By the definition of the symplectic form ω_p , we can deduce that d , which is determined by Eq. (49), is given by $d(f_i) = e_i^*$ and $d(f_i^*) = -e_i$, or in terms of matrix, we have

$$M(d) = \begin{pmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{pmatrix}.$$

By Corollary 4.14, there is a strict Lie 2-algebra $(\mathfrak{g}(\hat{A}), \hat{A}^*, d, [\cdot, \cdot]_s)$, where $[\cdot, \cdot]_s$ is given by the semidirect product Lie algebra structures on $\mathfrak{g}(\hat{A}) \ltimes_{L^*} \hat{A}^*$ associated to the left-symmetric algebra structure (55) on \hat{A} . Thus, $r = \sum_i (e_i \otimes f_i + e_i^* \otimes f_i^* - f_i \otimes e_i - f_i^* \otimes e_i^*)$ gives rise to a strict Lie 2-bialgebra structure on it. Note that this construction holds for any left-symmetric algebra without any constraint condition, which is regarded as a construction of strict Lie 2-bialgebras from the “twice double spaces” of left-symmetric algebras.

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